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Brane-bulk duality and non-conformal gauge theories

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Abstract: We discuss non-conformal gauge theories from type-IIB D3-branes embedded in orbifolded space-times. Such theories can be obtained by allowing some non-vanishing logarithmic twisted tadpoles. In certain cases with $\mathcal{N} = 0, 1$ supersymmetry correlation functions in the planar limit are the same as in the parent $\mathcal{N} = 2$ supersymmetric theories. In particular, the effective action in such theories perturbatively is not renormalized beyond one loop in the planar limit. In the $\mathcal{N} = 2$ as well as such $\mathcal{N} = 0, 1$ theories quantum corrections in the D3-brane gauge theories are encoded in the corresponding classical higher dimensional field theories whose actions contain the twisted fields with non-vanishing tadpoles. We argue that this duality can be extended to the non-perturbative level in the $\mathcal{N} = 2$ theories. We give some evidence that this might also be the case for $\mathcal{N} = 0, 1$ theories as well.

Keywords: 1/N Expansion, D-branes, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

In 't Hooft's large-N limit [1] gauge theories are expected to be drastically simplified. Thus, in this limit the gauge theory diagrams are organized in terms of Riemann surfaces, where each extra handle on the surface suppresses the corresponding diagram by $1/N^2$. The large-N expansion, therefore, resembles perturbative expansion in string theory. In the case of four-dimensional gauge theories this connection can be made precise in the context of type-IIB string theory in the presence of a large number N of D3-branes [2].[1] Thus, we consider a limit where $\alpha' \to 0$, $g_s \to 0$ and $N \to \infty$, while keeping $\lambda \equiv N g_s$ fixed, where g_s is the type-IIB string coupling. Note that in this context a world-sheet with g handles and b boundaries is weighted with

$$(N g_s)^b g_s^{2g-2} = \lambda^{2g-2+b} N^{-2g+2} \,. \tag{1.1}$$

Once we identify $g_s = g_{\rm YM}^2$, this is the same as the large-N expansion considered by 't Hooft. Note that for this expansion to make sense we must keep λ at a small value $\lambda < 1$. In this regime we can map the string diagrams directly to (various sums of) large-N Feynman diagrams. Note, in particular, that the genus $g = 0$ planar diagrams dominate in the large-N limit.[2]

If the space transverse to the D3-branes in the setup of [2] is $\mathbb{R}^6$, then we obtain the $\mathcal{N} = 4$ supersymmetric U(N) gauge theory on the D3-branes, which is conformal. On the other hand, we can also consider orbifolds of $\mathbb{R}^6$, which leads to gauge theories with reduced supersymmetry.[3] As was shown in [2], if we cancel all twisted tadpoles in such models, in the large-N limit the corresponding $\mathcal{N} = 0, 1, 2$ gauge theories are conformal. Moreover, in the planar limit the (on-shell) correlation functions in such theories are the same as in the parent $\mathcal{N} = 4$ gauge theory.

In this paper we discuss non-conformal gauge theories within the setup of [2]. Such theories can be obtained by allowing some twisted tadpoles to be non-vanishing. In particular, we can have consistent embeddings of non-conformal gauge theories if we allow logarithmic tadpoles, which correspond to the twisted sectors with fixed point loci of real dimension two. In particular, even though the corresponding string backgrounds are not finite (in the sense that we have logarithmic ultra-violet divergences), they are still consistent as far as the gauge theories are concerned, and the divergences correspond to the running in the four-dimensional gauge theories on the D3-branes.

Regularization of the aforementioned divergences can be conveniently discussed in the context of what we refer to as the brane-bulk duality, which is a consequence of the open-closed string duality. In particular, in certain non-trivial $\mathcal{N} = 0, 1$ cases in the planar limit the corresponding gauge theories perturbatively are not renormalized beyond one-loop. In fact, in this limit the (on-shell) correlation functions in these theories are the same as in the parent $\mathcal{N} = 2$ non-conformal gauge theories. In the $\mathcal{N} = 2$ as well as the aforementioned $\mathcal{N} = 0, 1$ cases the brane-bulk duality is particularly simple, and implies that the quantum corrections in the corresponding gauge theories are encoded in classical higher dimensional field theories whose

[1] The generalization of this setup in the presence of orientifold planes was discussed in [3].

[2] Note that if $\lambda > 1$, then no matter how large N is, for sufficiently many boundaries the higher genus terms become relevant, and we lose the genus expansion. In fact, in this regime one expects an effective supergravity description to take over as discussed in [4, 5, 6].

[3] The $\lambda > 1$ versions of these orbifold theories via the compactifications of type IIB on $AdS_5 \times (S^5/\Gamma)$ (where Γ is the orbifold group) were originally discussed in [7].

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actions contain the twisted fields with non-vanishing tadpoles. In particular, various quantum corrections can be obtained via integrating out the bulk fields in the corresponding classical action, that is, by considering the self-interaction of the D3-branes via the bulk fields. We give explicit computations in various $\mathcal{N} = 0, 1, 2$ examples in this context, including the treatment of divergences.

We also discuss whether the brane-bulk duality can be extended to the non-perturbative level in the aforementioned theories. In the $\mathcal{N} = 2$ cases we argue that, since we are working in the large-N limit, the low energy effective action does not receive non-perturbative corrections. We also conjecture that this should be the case for the corresponding $\mathcal{N} = 0, 1$ theories as well. In the $\mathcal{N} = 1$ cases we verify that there are no non-perturbative corrections to the superpotential in these theories in the large-N limit.

The remainder of this paper is organized as follows. In section 2 we discuss our setup. In section 3 we discuss non-conformal large-N gauge theories which can be constructed within this setup. In section 4 we discuss the large-N limit and brane-bulk-duality. In sections 5, 6 and 7 we give details of classical computations that in the context of the brane-bulk duality reproduce quantum results in the corresponding $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ gauge theories, respectively. In section 8 we comment on the non-perturbative extension of the brane-bulk duality. In section 9 we give a few concluding remarks. In appendix A we compute the brane-bulk couplings used in sections 5, 6 and 7.

2. Setup

In this section we discuss the setup within which we will consider four-dimensional large-N gauge theories in the context of brane-bulk duality. Parts of our discussion in this section closely follow [2]. Thus, consider type-IIB string theory in the presence of N coincident D3-branes with the space transverse to the D-branes $\mathcal{M} = \mathbb{R}^6/\Gamma$. The orbifold group $\Gamma = \{g_a | a = 1, \ldots, |\Gamma|\}$ ($g_1 = 1$) must be a finite discrete subgroup of Spin(6). If $\Gamma \subset \mathrm{SU}(3)(\mathrm{SU}(2))$, we have $\mathcal{N} = 1$ ($\mathcal{N} = 2$) unbroken supersymmetry, and $\mathcal{N} = 0$, otherwise.

We will confine our attention to the cases where type-IIB on $\mathcal{M}$ is a modular invariant theory.[4] The action of the orbifold on the coordinates X_i ($i = 1, \ldots, 6$) on $\mathcal{M}$ can be described in terms of SO(6) matrices: $g_a : X_i \rightarrow (g_a)_{ij} X_j$. The world-sheet fermionic superpartners of X_i transform in the same way. We also need to specify the action of the orbifold group on the Chan-Paton charges carried by the D3-branes. It is described by $N \times N$ matrices γ_a that form a representation of Γ. Note that γ_1 is an identity matrix, and $\mathrm{Tr}(\gamma_1) = N$.

[4]This is always the case if there are some unbroken supersymmetries. If all supersymmetries are broken, this is also true if $\nexists \mathbb{Z}_2 \subset \Gamma$. If $\exists \mathbb{Z}_2 \subset \Gamma$, then modular invariance requires that the set of points in $\mathbb{R}^6$ fixed under the $\mathbb{Z}_2$ twist has real dimension 2.

The D-brane sector of the theory is described by an oriented open string theory. In particular, the world-sheet expansion corresponds to summing over oriented Riemann surfaces with arbitrary genus g and arbitrary number of boundaries b, where the boundaries of the world-sheet are mapped to the D3-brane world-volume. Moreover, we must consider various "twists" around the cycles of the Riemann surface. The choice of these "twists" corresponds to a choice of homomorphism of the fundamental group of the Riemann surface with boundaries to Γ.

For example, consider the one-loop vacuum amplitude ($g = 0$, $b = 2$). The corresponding graph is an annulus whose boundaries lie on D3-branes. The annulus amplitude is given by

$$\mathcal{C} = \int_0^\infty \frac{dt}{t}\, \mathcal{Z}\,. \tag{2.1}$$

The one-loop partition function $\mathcal{Z}$ in the light-cone gauge is given by

$$\mathcal{Z} = \frac{1}{|\Gamma|} \sum_a \mathrm{Tr} \left(g_a\, \frac{1-(-1)^F}{2}\, e^{-2\pi t L_0} \right), \tag{2.2}$$

where L_0 is the light-cone hamiltonian, F is the fermion number operator, t is the real modular parameter on the cylinder, and the trace includes sum over the Chan-Paton factors. The states in the Neveu-Schwarz (NS) sector are space-time bosons and enter the partition function with weight $+1$, whereas the states in the Ramond (R) sector are space-time fermions and contribute with weight -1.

The elements g_a acting in the Hilbert space of open strings act both on the left end and the right end of the open string. This action corresponds to $\gamma_a \otimes \gamma_a$ acting on the Chan-Paton indices. The partition function (2.2), therefore, has the following form:

$$\mathcal{Z} = \frac{1}{|\Gamma|} \sum_a \left(\mathrm{Tr}(\gamma_a)\right)^2 \mathcal{Z}_a\,, \tag{2.3}$$

where $\mathcal{Z}_a$ are characters corresponding to the world-sheet degrees of freedom. The "untwisted" character $\mathcal{Z}_1$ is the same as in the $\mathcal{N} = 4$ theory for which $\Gamma = \{1\}$. The information about the fact that the orbifold theory has reduced supersymmetry is encoded in the "twisted" characters $\mathcal{Z}_a$, $a \neq 1$.

2.1 Tadpole cancellation

In this subsection we discuss one-loop tadpoles arising in the above setup. As was pointed out in [2], if all tadpoles are canceled, then the resulting theory is finite in the large-N limit. However, not all tadpoles need to be canceled to have a consistent four-dimensional gauge theory. In fact, we can obtain non-conformal gauge theories if we allow such tadpoles.

The characters $\mathcal{Z}_a$ in (2.3) are given by

$$\mathcal{Z}_a = \frac{1}{(8\pi^2\alpha' t)^2} \frac{1}{\left[\eta(e^{-2\pi t})\right]^{2+d_a}} \left[\mathcal{X}_a(e^{-2\pi t}) - \mathcal{Y}_a(e^{-2\pi t})\right], \tag{2.4}$$

where d_a is the real dimension of the set of points in $\mathbb{R}^6$ fixed under the twist g_a. The factor of $(8\pi^2\alpha' t)^2$ in the denominator comes from the bosonic zero modes corresponding to four directions along the D3-brane world-volume. Two of the η-functions come from the bosonic oscillators corresponding to two spatial directions along the D3-brane world-volume (the time-like and longitudinal contributions are absent as we are working in the light-cone gauge). The other d_a η-functions come from the bosonic oscillators corresponding to the directions transverse to the D-branes untouched by the orbifold twist g_a. Finally, the characters $\mathcal{X}_a$, $\mathcal{Y}_a$ correspond to the contributions of the world-sheet fermions, as well as the world-sheet bosons with g_a acting non-trivially on them (for $a \neq 1$):

$$\begin{aligned} \mathcal{X}_a &= \frac{1}{2} \operatorname{Tr}' \left[g_a e^{-2\pi t L_0} \right] , \\ \mathcal{Y}_a &= \frac{1}{2} \operatorname{Tr}' \left[g_a (-1)^F e^{-2\pi t L_0} \right] , \end{aligned} \tag{2.5}$$

where prime in Tr' indicates that the trace is restricted as described above.

For the annulus amplitude we therefore have

$$\mathcal{C} = \frac{1}{(8\pi^2\alpha')^2} \frac{1}{|\Gamma|} \sum_a \left[\mathcal{A}_a - \mathcal{B}_a \right] , \tag{2.6}$$

where

$$\begin{aligned} \mathcal{A}_a &= \left(\operatorname{Tr}(\gamma_a)\right)^2 \int_0^\infty \frac{dt}{t^3} \frac{1}{\left[\eta(e^{-2\pi t})\right]^{2+d_a}} \mathcal{X}_a(e^{-2\pi t}) , \\ \mathcal{B}_a &= \left(\operatorname{Tr}(\gamma_a)\right)^2 \int_0^\infty \frac{dt}{t^3} \frac{1}{\left[\eta(e^{-2\pi t})\right]^{2+d_a}} \mathcal{Y}_a(e^{-2\pi t}) . \end{aligned} \tag{2.7}$$

These integrals[5] are generically divergent as $t \to 0$ reflecting the presence of tadpoles. To extract these divergences we can change variables $t = 1/\ell$ so that the divergences correspond to $\ell \to \infty$:

$$\mathcal{A}_a = \left(\operatorname{Tr}(\gamma_a)\right)^2 \int_0^\infty \frac{d\ell}{\ell^{d_a/2}} \sum_{\sigma_a} N_{\sigma_a} e^{-2\pi \ell \sigma_a} , \tag{2.8}$$

$$\mathcal{B}_a = \left(\operatorname{Tr}(\gamma_a)\right)^2 \int_0^\infty \frac{d\ell}{\ell^{d_a/2}} \sum_{\rho_a} N_{\rho_a} e^{-2\pi \ell \rho_a} . \tag{2.9}$$

The closed string states contributing to $\mathcal{A}_a$ ($\mathcal{B}_a$) in the transverse channel are the NS-NS (R-R) states with $L_0 = \overline{L}_0 = \sigma_a(\rho_a)$ (and $N_{\sigma_a}(N_{\rho_a}) > 0$ is the number of such

[5] For space-time supersymmetric theories the total tadpoles vanish: $\mathcal{A}_a - \mathcal{B}_a = 0$. (The entire partition function vanishes as the numbers of space-time bosons and fermions are equal.) For consistency, however, we must extract tadpoles from individual contributions $\mathcal{A}_a$ and $\mathcal{B}_a$. Thus, for instance, cancellation of certain tadpoles coming from $\mathcal{B}_a$ is required for consistency of the equations of motion for the twisted R-R four-form which couples to D3-branes (see below).

states). The massive states with $\sigma_a(\rho_a) > 0$ do not lead to divergences as $\ell \to \infty$. On the other hand, the divergence property of the above integrals in the $\ell \to \infty$ limit is determined by the value of d_a. Given the orientability of Γ the allowed values of d_a are $0, 2, 4, 6$. For $d_1 = 6$ there is no divergence in $\mathcal{B}_1$, so we have no restriction for $\mathrm{Tr}(\gamma_1) = N$. For $d_a = 4$ the corresponding twisted NS-NS closed string sector contains tachyons. This leads to a tachyonic divergence in $\mathcal{A}_a$ unless $\mathrm{Tr}(\gamma_a) = 0$ for the corresponding g_a twisted sector. Finally, the ground states in the R-R sectors are massless, so we get divergences due to massless R-R states in the integral in $\mathcal{B}_a$ for large ℓ for $d_a = 0, 2$. In such sectors in non-supersymmetric cases we can also have tachyonic NS-NS divergences, while in supersymmetric cases we have massless NS-NS divergences.

To avoid complications with tachyons, for now we will focus on supersymmetric theories (we will discuss non-supersymmetric cases in section 7). We must therefore consider massless tadpoles arising for $d_a = 0, 2$. For $d_a = 0$ the corresponding integrals are linearly divergent with ℓ as $\ell \to \infty$. To cancel such a tadpole we must require that $\mathrm{Tr}\,(\gamma_a) = 0$ for the corresponding g_a twisted sector. On the other hand, if such a tadpole is not canceled, in the four-dimensional field theory language this would correspond to having a *quadratic* (in the momentum) divergence at the one-loop order. This would imply that the corresponding four-dimensional background is actually inconsistent in the sense that either four-dimensional supersymmetry[6] and/or Poincaré invariance must be broken.[7] This is related to the fact that in such cases the corresponding twisted closed string states, which propagate only in the D3-brane world-volume as they are supported at the orbifold fixed *points* in $\mathbb{R}^6$, have inconsistent field equations (yet they couple to the gauge/matter fields describing the low energy limit of the D3-brane world-volume theory). In fact, as was pointed out in [2], generically uncancelled tadpoles arising in the $d_a = 0$ cases lead to non-abelian gauge *anomalies* in the corresponding D3-brane gauge theories.
Thus, let us consider the following example. Let $\mathcal{M} = \mathbb{C}^3/\Gamma$, where the action of $\Gamma \approx \mathbb{Z}_3$ on the complex coordinates z_α ($\alpha = 1, 2, 3$) on $\mathcal{M}$ is that of the Z-orbifold: $g : z_\alpha \to \omega z_\alpha$ (where g is the generator of Γ, and $\omega \equiv e^{2\pi i/3}$). Next, let us choose the representation of Γ when acting on the Chan-Paton charges as follows: $\gamma_g = \mathrm{diag}(\mathbb{1}_{N_1}, \omega \mathbb{1}_{N_2}, \omega^2 \mathbb{1}_{N_3})$ (where $N_1 + N_2 + N_3 = N$, and $\mathbb{1}_m$ is an $m \times m$ identity matrix). The massless spectrum of this model is the $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N_1) \otimes \mathrm{U}(N_2) \otimes \mathrm{U}(N_3)$ gauge theory with the matter consisting of chiral supermultiplets in the following representations:

$$3 \times (\mathbf{N}_1, \overline{\mathbf{N}}_2, \mathbf{1})\,, \qquad 3 \times (\mathbf{1}, \mathbf{N}_2, \overline{\mathbf{N}}_3)\,, \qquad 3 \times (\overline{\mathbf{N}}_1, \mathbf{1}, \mathbf{N}_3)\,. \tag{2.10}$$

[6]Note that if $\Gamma \subset \mathrm{SU}(3)$ contains a twist with $d_a = 0$, we have $\mathcal{N} = 1$ supersymmetry.

[7]Since the corresponding twisted fields σ_a propagate in four dimensions, in the presence of the tadpoles their equations of motion read $\partial^\mu \partial_\mu \sigma_a = c_a + \cdots$ ($\mu = 0, 1, 2, 3$, c_a are the tadpoles, and the ellipses stand for terms of higher power in σ_a), so that we can *a priori* have solutions with broken four-dimensional Poincaré invariance.

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Note that this spectrum is anomalous for non-vanishing N_1, N_2, N_3 (the non-abelian gauge anomaly does not cancel) unless $N_1 = N_2 = N_3$. On the other hand, $\mathrm{Tr}(\gamma_g) = 0$ if and only if $N_1 = N_2 = N_3$. Here we should mention that not all the choices of γ_a that do not satisfy $\mathrm{Tr}(\gamma_a) = 0$ for $d_a = 0$ lead to such apparent inconsistencies. Thus, consider the same Γ as above but with $\gamma_g = \mathbb{1}_N$. The massless spectrum of this model is the $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N)$ gauge theory with no matter, so it is anomaly free. In fact, for any orbifold group Γ containing a twist g_a with $d_a = 0$ we obtain an anomaly free theory only if $\mathrm{Tr}(\gamma_a) = 0$ or $\gamma_a = \mathbb{1}_N$. In the latter case, however, as we discussed above, the corresponding background is nonetheless inconsistent, so we must require that for such twists $\mathrm{Tr}(\gamma_a) = 0$.

Finally, let us discuss $d_a = 2$ cases. For such twists the corresponding integrals are only logarithmically divergent as $\ell \to \infty$. If such a tadpole is not canceled, that is, if the corresponding $\mathrm{Tr}(\gamma_a) \neq 0$, in the four-dimensional field theory language this corresponds to having a *logarithmic* divergence (in the momentum) at the one-loop order. As we will see in the following, these logarithmic divergences are precisely related to the running in the corresponding gauge theories, which are *not* conformal (even in the large-N limit). Note that the corresponding twisted closed string states now propagate in two extra dimensions, so that the four-dimensional backgrounds are perfectly consistent — the tadpoles for these fields simply imply that these fields have non-trivial (logarithmic) profiles in these two extra dimensions, while the four dimensions along the D-brane world-volume are still flat (and the four-dimensional supersymmetry is unbroken). In fact, the presence of such tadpoles does not introduce any anomalies.[8] A simple way to see this for general Γ is to note that (in supersymmetric cases which we are focusing on here) an individual twist with $d_a = 2$ preserves $\mathcal{N} = 2$ supersymmetry, so that the corresponding gauge theory is anomaly free. This then immediately implies that we do not have any non-abelian gauge anomalies in a theory with multiple such twists either. Indeed, such anomalies would have to arise at the one-loop level. In the string theory language the relevant diagram is a $g = 0$, $b = 2$ diagram (the annulus) with three external lines corresponding to non-abelian gauge fields attached to a single boundary. Thus, if we attach two external lines to one boundary, while the third one to another boundary, the corresponding diagram will vanish — indeed, the Chan-Paton structure of such a diagram is given by (λ_r, $r = 1, 2, 3$, are the Chan-Paton matrices corresponding to the external lines)

$$\mathrm{Tr}\left(\lambda_1 \lambda_2 \gamma_a\right) \mathrm{Tr}\left(\lambda_3 \gamma_a\right), \tag{2.11}$$

which vanishes as for non-abelian gauge fields $\mathrm{Tr}(\lambda_r) = 0$, so $\mathrm{Tr}(\lambda_r \gamma_a) = 0$ as well for by definition λ_r are invariant under the orbifold group action as λ_r correspond

[8]This was shown for $\Gamma \approx \mathbb{Z}_m \otimes \mathbb{Z}_n$ in [8] in a somewhat more complicated way.

to the gauge bosons of the gauge group left unbroken by the orbifold (in particular, note that λ_r commute with γ_a). As to the aforementioned diagram with all three external lines attached to one boundary, its Chan-Paton structure is given by

$$\mathrm{Tr}\,(\lambda_1\lambda_2\lambda_3\gamma_a)\,\mathrm{Tr}\,(\gamma_a)\,. \tag{2.12}$$

For twists with $d_a = 2$, each diagram of this type is $\mathcal{N} = 2$ supersymmetric as it does not contain any information about further supersymmetry breaking (that is, the characters corresponding to the world-sheet degrees of freedom multiplying the Chan-Paton trace (2.12) are those of an $\mathcal{N} = 2$ theory), so such diagrams do not introduce any anomalies. We therefore conclude that a theory with multiple $d_a = 2$ twists with $\mathrm{Tr}(\gamma_a) \neq 0$ is also anomaly free.

Thus, as we see, in supersymmetric cases we can have non-trivial twists with $d_a = 0, 2$. Consistency of the background requires that

$$\mathrm{Tr}(\gamma_a) = 0 \qquad \text{for } d_a = 0\,, \tag{2.13}$$

while for $d_a = 2$ we do not have such a restriction.

3. Non-conformal large-N gauge theories

In this section we discuss certain large-N gauge theories arising in the above setup with some twists g_a with $d_a = 2$ such that $\mathrm{Tr}(\gamma_a) \neq 0$. As we have already mentioned, such theories are not conformal. The simplest examples of such theories are those with $\mathcal{N} = 2$ supersymmetry. Perturbatively such theories are not renormalized beyond the one-loop order. We can also construct non-conformal $\mathcal{N} = 1$ supersymmetric theories in this way. In general such theories are rather complicated. However, as we will see in the following, certain non-trivial $\mathcal{N} = 1$ theories of this type have the property that in the large-N limit the leading (that is, planar) diagrams do not renormalize the gauge theory correlators beyond the one-loop order. In fact, the corresponding correlation functions in such an $\mathcal{N} = 1$ theory are (up to overall numerical coefficients) the same as in a parent $\mathcal{N} = 2$ theory.

The simplest example of an $\mathcal{N} = 2$ theory is obtained if we take $\mathcal{M} = \mathbb{C}^3/\Gamma$, where the action of $\Gamma \approx \mathbb{Z}_2$ on the complex coordinates z_α ($\alpha = 1, 2, 3$) on $\mathcal{M}$ is as follows: $R : z_1 \to z_1$, $R : z_{2,3} \to -z_{2,3}$ (where R is the generator of Γ). Next, let us choose the representation of Γ when acting on the Chan-Paton charges as follows: $\gamma_R = \mathbb{1}_N$. The massless spectrum of this model is the $\mathcal{N} = 2$ supersymmetric $\mathrm{U}(N)$ gauge theory with no matter. The non-abelian part of this theory is not conformal, and is asymptotically free. As to the overall center-of-mass $\mathrm{U}(1)$, it is free, and can therefore be ignored.

More generally, if we have twists with $\gamma_a \neq \mathbb{1}_N$, the gauge group generically is a product of $\mathrm{U}(N_k)$ factors, and we also have matter, which can be obtained using the quiver diagrams (see [9, 10, 11][9]). There is always an overall center-of-mass U(1), which is free. Other U(1) factors, however, run as the matter is charged under them. In the large-N limit, however, these U(1)'s decouple in the infra-red, and can therefore also be ignored.[10]

3.1 Non-renormalization theorems in $\mathcal{N}=1$ theories

As we have already mentioned, within the above setup we can construct non-trivial $\mathcal{N}=1$ supersymmetric theories which are not renormalized beyond the one-loop order in the planar limit. The idea here is similar to that in [2], where it was noticed that in theories with vanishing twisted tadpoles the planar diagrams are the same (up to overall numerical factors) as in the parent $\mathcal{N}=4$ theory. This is because the diagrams that contain information about supersymmetry breaking always contain twisted Chan-Paton traces $\mathrm{Tr}(\gamma_a)$, which vanish in such theories. In this subsection we will consider theories where the planar diagrams are the same as in a parent $\mathcal{N}=2$ theory in essentially the same way.

To obtain such models, consider an orbifold group Γ, which is a subgroup of SU(3) but is not a subgroup of SU(2). Let $\widetilde{\Gamma}$ be a non-trivial subgroup of Γ such that $\widetilde{\Gamma} \subset \mathrm{SU}(2)$. We will allow the Chan-Paton matrices γ_a corresponding to the twists $g_a \in \widetilde{\Gamma}$ (which have $d_a = 2$) not to be traceless, so that the corresponding $\mathcal{N}=2$ model is not conformal. However, we will require that the other Chan-Paton matrices γ_a for the twists $g_a \notin \widetilde{\Gamma}$ be traceless. The resulting $\mathcal{N}=1$ model is not conformal. However, in the planar limit the perturbative gauge theory amplitudes are not renormalized beyond one loop.

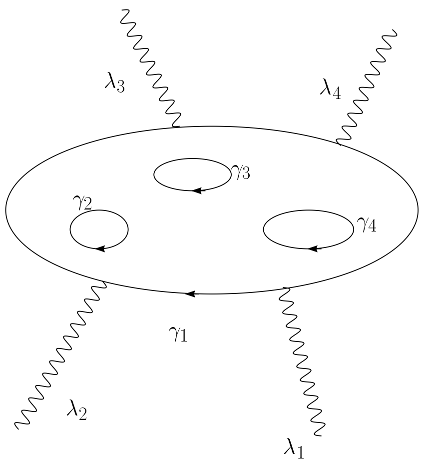

Figure 1: A planar diagram.

The proof of this statement is straightforward. Thus, consider a planar diagram with b boundaries (but no handles)[11] with all external lines attached to a single boundary, which without loss of generality can be chosen to be the outer boundary. Such a diagram is depicted in figure 1. Next, we

[9] A T-dual description of such models can be studied in the context of the brane-box models [12].

[10] As was pointed out in [13, 14], in some cases these U(1)'s are actually anomalous, and are broken at the tree-level via the generalized Green-Schwarz mechanism. In particular, in cases with twists with $d_a = 0$ we have mixed $\mathrm{U}(1)_k\, \mathrm{SU}(N_l)^2$ anomalies. However, in the cases with matter we are interested in here we do have running U(1)'s that are anomaly free. These U(1)'s decouple in the infra-red in the large-N limit.

[11] Such a diagram corresponds to a $(b-1)$-loop diagram in the field theory language.

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need to specify the twists on the boundaries. A convenient choice (consistent with that made for the annulus amplitude (2.2)) is given by[12]

$$\gamma_{a_1} = \prod_{s=2}^{b} \gamma_{a_s} \,, \tag{3.1}$$

where γ_{a_1} corresponds to the outer boundary, while γ_{a_s}, $s = 2, \ldots, b$, correspond to inner boundaries. Let λ_r, $r = 1, \ldots, M$, be the Chan-Paton matrices corresponding to the external lines. Then the above planar diagram has the following Chan-Paton group-theoretic dependence:

$$\sum \mathrm{Tr}\left(\gamma_{a_1}\lambda_1 \cdots \lambda_M\right) \prod_{s=2}^{b} \mathrm{Tr}(\gamma_{a_s}) \,, \tag{3.2}$$

where the sum involves all possible distributions of the γ_{a_s} twists that satisfy the condition (3.1), as well as permutations of the λ_r factors (note that λ_r here correspond to the states that are kept after the orbifold projections, so that they commute with all γ_a). Note that the diagrams with all twists $g_a \in \widetilde{\Gamma}$ are (up to overall numerical coefficients) the same as in the parent $\mathcal{N} = 2$ theory, and therefore vanish beyond one loop. All other diagrams contain at least one twist $g_{a_s} \notin \widetilde{\Gamma}$ with $s > 1$. This follows from the condition (3.1). This then implies that all such diagrams vanish as

$$\mathrm{Tr}(\gamma_a) = 0 \,, \qquad g_a \notin \widetilde{\Gamma} \,. \tag{3.3}$$

In particular, this implies that the non-abelian gauge couplings do not run in the large-N limit beyond one loop.[13] The anomalous scaling dimensions (two-point functions corresponding to the wave-function renormalization) for matter fields,[14] the corresponding diagrams are (up to overall numerical factors) the same as in the parent $\mathcal{N} = 2$ theory, so they are not renormalized in the planar limit. Finally, the Yukawa (three-point) and quartic scalar (four-point) couplings are not renormalized in perturbation theory due to the $\mathcal{N} = 1$ non-renormalization theorem for the superpotential.

Let us now consider a simple example of the aforementioned gauge theories. Thus, let $\widetilde{\Gamma} = \{\widetilde{g}_a | a = 1, \ldots, |\widetilde{\Gamma}|\} \subset \mathrm{SU}(2)$, and let the corresponding twisted Chan-Paton matrices $\widetilde{\gamma}_a = \mathbb{1}_{2N}$. Let us assume that $\widetilde{g}_a$ act non-trivially on the complex coordinates z_2, z_3 on $\widetilde{\mathcal{M}} = \mathbb{C}^3/\widetilde{\Gamma}$, while leaving the coordinate z_1 untouched. Moreover,

[12]Here some care is needed in the cases where the orbifold group Γ is non-abelian, and we have to choose base points on the world-sheet to define the twists. Our discussion here, however, is unmodified also in this case.

[13]More precisely, the higher loop contributions to the gauge coupling running, which come from the diagrams with handles, are subleading in the large-N limit compared with the leading one-loop contribution. This is analogous to what happens in theories discussed in [15]. In fact, the techniques used there to prove that the higher loop corrections are subleading are very similar to the one we are using here.

[14]Note that in such theories all matter fields are non-trivially charged under the non-abelian gauge subgroup.

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let us assume that the orbifold group $\widetilde{\Gamma}$ is abelian, and its action on the coordinates z_2, z_3 is diagonal. Next, let R be a generator of a $\mathbb{Z}_2$ group with the following action: $R : z_{1,2} \to -z_{1,2}$, $R : z_3 \to z_3$. Note that R commutes with $\widetilde{g}_a \in \widetilde{\Gamma}$. The full orbifold group is given by $\Gamma = \{\widetilde{g}_a, \widehat{g}_a | a = 1, \dots, |\widetilde{\Gamma}|\}$, which is also abelian, and $\widehat{g}_a \equiv R\widetilde{g}_a$. Let the twisted Chan-Paton matrix $\gamma_R = \mathrm{diag}(\mathbb{1}_N, -\mathbb{1}_N)$. Then the theory is the $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N) \otimes \mathrm{U}(N)$ gauge theory with chiral matter multiplets, call them ϕ and χ, in $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$, respectively (in the following we will ignore the U(1) factors). Note that there is no tree-level superpotential in this theory:

$$\mathcal{W}_{\mathrm{tree}} = 0\,. \tag{3.4}$$

This implies that we can break the gauge group down to the diagonal $\mathrm{SU}(N)$ subgroup by giving the appropriate vacuum expectation values to the fields ϕ and χ. The resulting theory in the IR is then the $\mathcal{N} = 2$ supersymmetric $\mathrm{SU}(N)$ gauge theory without matter. In the string theory language this corresponds to moving the D3-branes off the $\mathbb{Z}_2$ orbifold singularity, hence the $\mathcal{N} = 2$ supersymmetry of the resulting theory.

3.2 Pure supergluodynamics

Before we end this section, as an aside we would like to discuss an embedding of pure $\mathcal{N} = 1$ supergluodynamics in the setup discussed in section 2. Thus, consider the orbifold group $\Gamma \approx \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the generators R_1 and R_2 of the two $\mathbb{Z}_2$'s acting on the complex coordinates z_α on $\mathcal{M} = \mathbb{C}^3/\Gamma$ as follows:

$$R_\alpha : z_\beta \to -(-1)^{\delta_{\alpha\beta}} z_\beta\,, \tag{3.5}$$

where $R_3 \equiv R_1 R_2$. Next, let the twisted Chan-Paton matrices be given by $\gamma_{R_\alpha} = \mathbb{1}_N$. The D3-brane gauge theory is then given by the $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(N)$ super-Yang-Mills theory without matter (plus an overall U(1) which can be ignored). Note that the real dimensions of the points fixed under the twists R_α are $d_\alpha = 2$.

The above embedding shows that at least perturbatively pure supergluodynamics possesses a discrete symmetry, which is not evident in the field theory language. Thus, we have the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ symmetry as all the interactions involving twisted closed string states must be invariant under the orbifold group. In addition, we have a $\mathbb{Z}_3$ symmetry which permutes the closed string states coming from the R_α twisted sectors. In particular, the generator θ of this $\mathbb{Z}_3$ group has the following action:

$$\theta R_\alpha \theta^{-1} = R_{\alpha+1}\,, \qquad R_{\alpha+3} \equiv R_\alpha\,. \tag{3.6}$$

Thus, the $\mathbb{Z}_3$ subgroup does not commute with the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ subgroup. Thus, R_1 and θ generate a non-abelian discrete subgroup of SO(3), namely, the *tetrahedral* group $\mathcal{T}$.

Thus, as we see, the perturbative large-N pure superglue theory possesses a discrete $\mathbb{Z}_3$ symmetry. In fact, this symmetry persists even at finite N. Indeed, even

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the diagrams with handles possess this symmetry as this symmetry is a symmetry of the underlying embedding of the pure supergluodynamics into string theory via the orbifold setup. In fact, within this setup the $\mathbb{Z}_3$ symmetry is a discrete *gauge* symmetry. Indeed, instead of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, the orbifold group can be chosen as the full tetrahedral group $\mathcal{T}$, where the generator θ of the $\mathbb{Z}_3$ subgroup of $\mathcal{T}$ acts on the complex coordinates z_α as follows: $\theta : z_\alpha \to z_{\alpha+1}$ ($z_{\alpha+3} \equiv z_\alpha$). If we now choose the twisted Chan-Paton matrix $\gamma_\theta = \mathbb{1}_N$, then the D3-brane gauge theory is still pure superglue. This embedding makes it evident that the $\mathbb{Z}_3$ symmetry is indeed a gauge symmetry, so it might be an exact symmetry of pure superglue even non-perturbatively.[15]

We would like to end this subsection by pointing out one immediate consequence of the aforementioned $\mathbb{Z}_3$ symmetry — both the one-loop as well as the two-loop[16] β-function coefficients in the pure supergluodynamics are multiples of 3.

4. Large-N limit and brane-bulk duality

As we mentioned in Introduction, perturbative expansion of D3-brane gauge theories simplifies substantially in the large-N limit. It is advantageous to work in the full string theory framework, which in a way is much simpler than the Feynman diagram techniques. (At the end of the day we will take the $\alpha' \to 0$ limit, which amounts to reducing the theory to the gauge theory subsector.) Thus, in the string theory language there are two classes of diagrams we need to consider: (i) diagrams without handles; (ii) diagrams with handles. The latter correspond to closed string loops, and can be neglected as they are subleading in the large-N limit. We will therefore focus on the diagrams without handles. The latter diagrams can be viewed as *tree-level* closed string graphs connecting various boundaries. This suggests that (upon taking the $\alpha' \to 0$ limit) in some cases (which we will identify in a moment) we might be able to rewrite the perturbative expansion in the large-N limit of the corresponding gauge theories in the language where various quantum corrections in the gauge theory are encoded in a *classical* higher dimensional *field* theory. The effective quantum action for the four-dimensional large-N gauge theory is then obtained by starting with the corresponding higher dimensional classical action, integrating out the bulk fields, and restricting to the gauge theory subsector.[17] That is, we consider the D-branes

[15] We can identify other discrete gauge symmetries of this theory in a similar fashion. Thus, consider an orbifold group Γ which is a subgroup of SO(3) but not a subgroup of SU(2). The corresponding gauge theory has $\mathcal{N} = 1$ supersymmetry. Note that the non-trivial twists in Γ all have $d_a = 2$, and we can choose the corresponding twisted Chan-Paton matrices as $\gamma_a = \mathbb{1}_N$, so that (upon dropping the overall U(1)) we obtain the SU(N) pure super-Yang-Mills theory. An appropriate abelian subgroup of Γ is then a symmetry of this theory.

[16] Note that beyond two loops the β-function becomes gauge dependent.

[17] This way we can obtain perturbative corrections to various *on-shell* gauge theory correlators as we are using an on-shell formulation of the corresponding string theory.

in the background of the bulk fields that is created by the D-branes themselves.[18] The field theory cut-off in this language arises precisely from the fact that, in the above setup, the twisted closed string states have tadpoles resulting in logarithmic profiles in the extra dimensions, and we need to regularize these profiles at the sources, that is, the D-branes, whose locations in the extra dimensions are given by points in $\mathcal{M}$.

4.1 Brane-bulk duality

The aforementioned proposal that in the large-N limit various quantum corrections in the D-brane gauge theory should in some cases be encoded in a higher dimensional classical field theory is what we refer to as the *brane-bulk duality*. In this subsection we would like to identify the cases where the brane-bulk duality is applicable as stated above. Note that the brane-bulk duality is a consequence of the *open-closed duality* property of string theory. In particular, the diagrams without handles in the transverse closed string channel can indeed be viewed as tree-level closed string graphs connecting various boundaries. At the one-loop order the corresponding diagram (up to external lines) is an annulus, where two boundaries are connected by a single closed string tube. In particular, this diagram does *not* involve closed string interactions. Moreover, in the field theory limit $\alpha' \to 0$ contributions due to the massive closed string states in the transverse closed string channel are always finite. That is, these contributions correspond to heavy string *thresholds*. In the field theory language they translate into subtraction scheme dependent artifacts, which arise due to a particular choice of regularization. Thus, at the one-loop order the contributions due to massive string modes can be absorbed into the subtraction scheme dependence, so that in the field theory limit the brane-bulk duality indeed holds — various one-loop corrections in the gauge theory can be computed by calculating the classical D-brane self-interaction via the massless bulk fields.

Beyond the one-loop order, however, that is, when the number of boundaries is greater than two, we must include closed string interactions in the transverse closed string channel. The massive closed string states are now expected to contribute in a non-trivial way. In particular, *a priori* there is no longer a clear interpretation of these contributions in terms of thresholds. Another way of phrasing this is that at higher loop orders we can have mixed contributions coming from both massless as well as massive states. That is, in general the brane-bulk duality can still be formulated except that at higher loop orders it will involve massless as well as *infinitely* many massive states as far as the corresponding higher dimensional classical "field theory" is concerned. In particular, in general this classical "field theory" is nothing but the corresponding closed string theory, so we are back to the original open-closed string duality.

[18] Such a self-interaction in the case of strings was discussed in [16, 17].

There are, however, non-trivial cases where in the large-N limit the bulk closed string theory can be consistently truncated to a field theory containing only a finite number of massless fields. These cases are those where in the planar limit the gauge theory perturbatively is not renormalized beyond one loop. Thus, this is clearly the case for $\mathcal{N} = 2$ theories. In theories without matter this, in fact, holds even at finite N. In theories with matter, however, we have running U(1)'s whose decoupling is ensured only in the large-N limit.

In subsection 3.1 we discussed $\mathcal{N} = 1$ theories which perturbatively are not renormalized beyond the one-loop order in the planar limit. It is clear that the aforementioned formulation of the brane-bulk duality also holds in such theories, where, once again, various quantum corrections in the gauge theory are encoded in a higher dimensional classical field theory with a finite number of massless fields. In fact, in $\mathcal{N} = 2$ as well as $\mathcal{N} = 1$ theories the bulk fields entering the relevant part of the corresponding higher dimensional classical action are *twisted* closed string states. Indeed, at the one-loop order the diagrams involving untwisted closed string states in the transverse closed string channel are $\mathcal{N} = 4$ supersymmetric, so that they do not contribute into the renormalization of the corresponding (on-shell) gauge theory correlators.

To clarify the above discussion, let us give a schematic description of the above procedure for obtaining quantum corrections in the corresponding gauge theories via the brane-bulk duality. Thus, we start from the classical action

$$S = S_{\mathrm{brane}}[\Phi] + S_{\mathrm{bulk}}[\sigma] + S_{\mathrm{int}}[\Phi, \sigma]\,, \tag{4.1}$$

where Φ is a collective notation for the (fractional) D3-brane world-volume fields, while σ is a collective notation for the *massless twisted bosonic* (NS-NS and R-R) bulk fields; $S_{\mathrm{brane}}[\Phi]$ is the classical four-dimensional action for the brane fields Φ, while $S_{\mathrm{bulk}}[\sigma]$ is the classical higher dimensional action for the bulk fields σ (note that some of these fields such as twisted closed string states propagate in less than 10 dimensions); finally, $S_{\mathrm{int}}[\Phi, \sigma]$, which is a classical four-dimensional functional (with the integral over the (fractional) D3-brane world-volume), describes the coupling of the bulk fields to the brane fields, as well as to the brane itself (in particular, it includes all tadpoles). Since here we are interested in one-loop corrections in the gauge theory language, the action $S_{\mathrm{bulk}}[\sigma]$ contains only the terms quadratic in σ (that is, the kinetic terms), while $S_{\mathrm{int}}[\Phi, \sigma]$ contains only the terms linear in σ. Moreover, the fields σ include only the twisted fields with non-vanishing (logarithmic) tadpoles.

Next, we solve the equations of motion for σ following from the above action:

$$\frac{\delta S_{\mathrm{bulk}}}{\delta\sigma} = -\frac{\delta S_{\mathrm{int}}}{\delta\sigma}\,. \tag{4.2}$$

In particular, we are interested in the solution where the fields σ do *not* explicitly depend on the coordinates x^μ along the D3-brane world-volume, but only functionally via $\Phi(x^\mu), \partial_\nu\Phi(x^\mu), \ldots$; this solution, however, does in general explicitly depend on the coordinates x^i transverse to the D-branes. Let us denote this solution via $\overline{\sigma}$. The effective quantum action for the fields Φ in the large-N limit is then given by

$$S^{\rm qu}_{\rm brane}[\Phi] = S_{\rm brane}[\Phi] + S_{\rm int}[\Phi, \overline{\sigma}] \, . \tag{4.3}$$

Note that $\overline{\sigma}|_{\rm D3}$ are generically divergent, so we need to further clarify the meaning of the second term in (4.3). Thus, let

$$J \equiv \left. \frac{\delta S_{\rm int}}{\delta \sigma} \right|_{\Phi = 0} . \tag{4.4}$$

Note that J are independent of x^μ, and, in fact, are nothing but the tadpoles for the fields σ. If all $\mathrm{Tr}(\gamma_a) = 0$ (for $d_a = 0, 2$), then there are no tadpoles for the corresponding twisted fields (that is, the corresponding $J = 0$), and the large-N gauge theory is finite [2].[19] On the other hand, if some $\mathrm{Tr}(\gamma_a) \neq 0$ for twists with $d_a = 2$, some divergences no longer cancel, and we need to regularize $\overline{\sigma}|_{\rm D3}$ in (4.3). Note, however, that the corresponding divergences are only logarithmic, and, in fact, correspond to the running in the four-dimensional gauge theory, which is no longer conformal.

4.2 A toy model

In this subsection we would like to address one technical point in the context of the brane-bulk duality, namely, the issue of regularizing the aforementioned divergences. Here we will consider a simple toy model which possesses the main ingredients for illustrating the regularization procedure. In fact, as we will see in the following sections, this model actually captures all the key features of the $\mathcal{N} = 2$ models as well the $\mathcal{N} = 1$ models discussed in subsection 3.1.

Thus, consider the six dimensional theory with the following action:

$$S = -\frac{1}{2} \int d^6x \, \partial_M \phi \, \partial^M \phi - a \int_\Sigma d^4x \, \mathcal{O}_4 - bL^2 \int_\Sigma d^4x \, \phi \, \mathcal{O}_4 - cL^{-2} \int_\Sigma d^4x \, \phi \, . \tag{4.5}$$

Here $\phi(x^M)$ is a six dimensional scalar field, whose dimension is $(\text{mass})^2$; $\mathcal{O}_4(x^\mu)$ is a four-dimensional operator with dimension $(\text{mass})^4$ localized on the hypersurface Σ; the couplings a, b, c (which are assumed to be non-vanishing) are dimensionless; finally, L is a parameter of dimension (length).

In the above action ϕ is an analog of a twisted closed string state with a non-vanishing tadpole (the last term in (4.5)); Σ plays the role of a 3-brane; $\mathcal{O}_4$ is an analog of a dimension-four gauge theory operator such as $\mathrm{Tr}(F^2)$; finally, L^2 is analogous to α'.

[19]More precisely, various U(1) factors can still run, but, as we have already mentioned, in the large-N limit they decouple in the infra-red, and can therefore be ignored.

Following the procedure described in the previous subsection, we look for a solution to the equation of motion for ϕ:

$$\partial_M \partial^M \phi = \left[bL^2 \mathcal{O}_4(y) + cL^{-2}\right] \delta^{(2)}(z) \,. \tag{4.6}$$

Here $z^i \equiv x^i$ are the coordinates transverse to the 3-brane (whose location is chosen to be $z^i = 0$); also, we use the notation $y^\mu \equiv x^\mu$. It is convenient to Fourier transform to the momentum space

$$\phi(p,k) \equiv \int d^4y\, d^2z\, e^{ip\cdot y + ik\cdot z}\, \phi(y,z) \,, \tag{4.7}$$

where p^μ and k^i are the momenta corresponding to y^μ and z^i, respectively. Thus, we have the following solution

$$\overline{\phi}(p,k) = -\left[bL^2 \frac{\mathcal{O}_4(p)}{k^2+p^2} + (2\pi)^4 cL^{-2} \frac{\delta^{(4)}(p)}{k^2}\right] . \tag{4.8}$$

This gives

$$\overline{\phi}(p,z) = -\int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot z} \left[bL^2 \frac{\mathcal{O}_4(p)}{k^2+p^2} + (2\pi)^4 cL^{-2} \frac{\delta^{(4)}(p)}{k^2}\right] . \tag{4.9}$$

The corresponding effective quantum action on the 3-brane is then given by

$$\begin{aligned} S_\Sigma^{\rm qu} &= -a \int_\Sigma d^4y\, \mathcal{O}_4(y) - bL^2 \int_\Sigma d^4y\, \overline{\phi}(y, z=0)\, \mathcal{O}_4(y) - cL^{-2} \int_\Sigma d^4y\, \overline{\phi}(y, z=0) \\ &= -a\mathcal{O}_4(p=0) - bL^2 \int \frac{d^4p}{(2\pi)^4}\, \overline{\phi}(p, z=0)\, \mathcal{O}_4(-p) - cL^{-2} \overline{\phi}(p=0, z=0) \\ &= -\left[a - \frac{bc}{2\pi^2} \int \frac{d^2k}{k^2}\right] \mathcal{O}_4(p=0) + b^2L^4 \int \frac{d^4p}{(2\pi)^4} \int \frac{d^2k}{(2\pi)^2} \frac{\mathcal{O}_4(p)\mathcal{O}_4(-p)}{k^2+p^2} + \cdots , \end{aligned} \tag{4.10}$$

where the ellipses in the last line stand for a divergent constant piece (which does not contain the operator $\mathcal{O}_4$, and is proportional to c^2 as it is solely due to the presence of the ϕ tadpole).

The second term in the last line in (4.10) is a non-local higher dimensional operator, and it disappears in the $L^2 \to 0$ limit. The first term containing $\mathcal{O}_4(p = 0)$ is the same as that in the classical 3-brane world-volume action except for the corresponding coupling — the quantum coupling in $S_\Sigma^{\rm qu}$ is a renormalized coupling:

$$\widetilde{a} = a - \frac{bc}{2\pi^2} \int_{k^2=\mu^2}^{k^2=\Lambda^2} \frac{d^2k}{k^2} = a - \frac{bc}{2\pi} \ln\left(\frac{\Lambda^2}{\mu^2}\right) . \tag{4.11}$$

Here Λ is an ultra-violet (UV) cut-off, while μ is the infra-red (IR) cut-off. In the four-dimensional 3-brane world-volume field theory language the IR cut-off is interpreted as the renormalization group (RG) scale at which the renormalized coupling $\widetilde{a} = \widetilde{a}(\mu)$ is measured.

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Thus, if in the above toy model we adapt the interpretation of the previous subsection, from the classical dynamics of the bulk field ϕ we will obtain the "one-loop" renormalization of the 3-brane world-volume field theory. In fact, in this model the 3-brane world-volume theory is not renormalized beyond the "one-loop" order.

5. Examples with $\mathcal{N} = 2$ supersymmetry

In this section we would like to apply the brane-bulk duality discussed in the previous section to gauge theories arising in the orbifold construction discussed in section 2. Here we will focus on the simplest examples of this type, in particular, those with $\mathcal{N} = 2$ supersymmetry. Perturbatively such gauge theories are not renormalized beyond the one-loop order even for finite N. Thus, the brane-bulk duality in such examples can be understood in detail.[20]

We described the simplest example of an $\mathcal{N} = 2$ gauge theory in the above context in the beginning of the previous section. In this example $\Gamma \approx \mathbb{Z}_2$, and the twisted Chan-Paton matrix $\gamma_R = \mathbb{1}_N$. The gauge group is $\mathrm{U}(N)$, and we have no matter fields. In subsection 5.1 we discuss the brane-bulk duality in this model in the case of unbroken gauge symmetry. We will discuss the case of spontaneously broken gauge symmetry in subsection 5.2. Finally, in subsection 5.3 we discuss examples with matter.

5.1 Unbroken gauge symmetry

To discuss the brane-bulk duality in this model, we need the relevant part of the classical action containing the D3-brane as well as the bulk fields. This action has the following form [9] (we are not including terms containing $F \wedge F$ for we are going to be interested in renormalization of the F^2 term):[21]

$$S = -\int_{\mathrm{D3}} \Big(a\, \mathrm{Tr}\,[F^{\mu\nu} F_{\mu\nu}] + b\, \sigma\, \mathrm{Tr}\,[F^{\mu\nu} F_{\mu\nu}] + c\, \sigma + d\, \epsilon^{\mu\nu\sigma\rho}\, C_{\mu\nu}\, \mathrm{Tr}\,[F_{\sigma\rho}] \Big) - \\ - \int_{\mathrm{D3}\times\mathbb{R}^2} \left(\frac{1}{2} \partial^\mu \sigma\, \partial_\mu \sigma + \frac{1}{12} H^{\mu\nu\sigma} H_{\mu\nu\sigma} \right). \tag{5.1}$$

Here σ is a twisted NS-NS scalar, while $C_{\mu\nu}$ is a twisted two-form (whose field strength is $H_{\mu\nu\sigma}$). We have normalized the kinetic terms of the bulk σ and $C_{\mu\nu}$ fields in the standard way. Once this normalization is fixed, the couplings b, c, d can be determined. We discuss these couplings in appendix A. The combinations relevant

[20] Here we are considering the regime where the effective 't Hooft coupling is small. Supergravity duals of $\mathcal{N} = 2$ theories in the regime where the effective 't Hooft coupling is large were discussed in [18, 19].

[21] Here we are using the fact that $\gamma_R = \mathbb{1}_N$. Also, in $\mathrm{Tr}\,[F_{\mu\nu}]$ only the $\mathrm{U}(1)$ contribution is non-vanishing.

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for our discussion here are given by

$$bc = 2Nd^2 = \frac{N}{8\pi}\,. \tag{5.2}$$

Finally, the coupling a is given by

$$a = \frac{1}{2g_{\rm YM}^2}\,, \tag{5.3}$$

where $g_{\rm YM}$ is the tree-level Yang-Mills gauge coupling, and the $\mathrm{U}(N)$ generators T^A are normalized as $2\,\mathrm{Tr}\left(T^A T^B\right) = \delta^{AB}$. The Yang-Mills gauge coupling is related to the string coupling g_s via $g_{\rm YM}^2 = 2\pi g_s$.

As in section 4, we solve classical equations of motion for the fields σ and $C_{\mu\nu}$ (the latter is a gauge field, so we must use gauge fixing such as $\partial^\mu C_{\mu\nu} = 0$), and integrate them out of the classical action (5.1). The resulting effective quantum action is given by (we are using the momentum representation, and drop higher dimensional terms as well as those independent of $F_{\mu\nu}$):

$$\begin{aligned} S^{\rm qu}_{\rm brane} = -\int \frac{d^4p}{(2\pi)^4}\Bigg(&\left[a - \frac{bc}{2\pi^2}\int\frac{d^2k}{k^2}\right]\mathrm{Tr}\left[F^{\mu\nu}(p)\;F_{\mu\nu}(-p)\right] + \\ &+\frac{d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\,\mathrm{Tr}\left[F^{\mu\nu}(p)\right]\;\;\mathrm{Tr}\left[F_{\mu\nu}(-p)\right]\Bigg)\,. \end{aligned} \tag{5.4}$$

Since only the $\mathrm{U}(1)$ subgroup contributes into $\mathrm{Tr}[F_{\mu\nu}]$, we can rewrite the last term in the above expression in terms of the $\mathrm{U}(1)$ field strength $f_{\mu\nu}$ (note that the corresponding generator is $\mathbb{1}_N/\sqrt{2N}$):

$$\mathrm{Tr}[F^{\mu\nu}(p)]\,\mathrm{Tr}[F_{\mu\nu}(-p)] = \mathrm{Tr}[f^{\mu\nu}(p)]\,\mathrm{Tr}[f_{\mu\nu}(-p)] = N\,\mathrm{Tr}[f^{\mu\nu}(p)\,f_{\mu\nu}(-p)]\,. \tag{5.5}$$

Thus, we have

$$\begin{aligned} S^{\rm qu}_{\rm brane} = -\int \frac{d^4p}{(2\pi)^4}\Bigg(&\left[a - \frac{bc}{2\pi^2}\int\frac{d^2k}{k^2}\right]\mathrm{Tr}\left[\widehat{F}^{\mu\nu}(p)\widehat{F}_{\mu\nu}(-p)\right] + \\ &+\left[a - \frac{bc}{2\pi^2}\int\frac{d^2k}{k^2} + \frac{Nd^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\right]\mathrm{Tr}\left[f^{\mu\nu}(p)\,f_{\mu\nu}(-p)\right]\Bigg)\,, \end{aligned} \tag{5.6}$$

where $\widehat{F}_{\mu\nu}$ is the $\mathrm{SU}(N)$ field strength. Note that the integrals contributing to the f^2 coupling individually are UV divergent. However, since we have (5.2), the total contribution is UV finite. On the other hand, the first integral is IR divergent, while the second integral is IR finite for $p^2 > 0$ (for this latter integral the p^2 term in the denominator plays the role of the IR cut-off). We must therefore introduce an IR cut-off in the first integral. The fact that the $\mathrm{U}(1)$ gauge coupling should not be renormalized then dictates that the IR cut-off in the first integral must be chosen as follows:[22]

$$a - \frac{bc}{2\pi^2}\int_{k^2=p^2}\frac{d^2k}{k^2} + \frac{Nd^2}{\pi^2}\int_{k^2=0}\frac{d^2k}{k^2+p^2} = a\,. \tag{5.7}$$

The effective quantum action therefore reads:

$$S^{\rm qu}_{\rm brane} = -\int\frac{d^4p}{(2\pi)^4}\left(\widetilde{a}(p^2)\,{\rm Tr}\left[\widehat{F}^{\mu\nu}(p)\,\widehat{F}_{\mu\nu}(-p)\right] + a\,{\rm Tr}\left[f^{\mu\nu}(p)\;f_{\mu\nu}(-p)\right]\right), \tag{5.8}$$

where

$$2\widetilde{a}(p^2) \equiv 2a - \frac{bc}{\pi^2}\int_{k^2=p^2}^{k^2=\Lambda^2}\frac{d^2k}{k^2} = \frac{1}{g^2_{\rm YM}} + \frac{\beta_0}{16\pi^2}\;\ln\left(\frac{\Lambda^2}{p^2}\right). \tag{5.9}$$

This is nothing but the one-loop renormalized Yang-Mills gauge coupling with the β-function coefficient $\beta_0 = -2N$.

5.2 Spontaneously broken gauge symmetry

In this subsection we will discuss the brane-bulk duality in the above model in the case where the gauge symmetry is spontaneously broken: $\mathrm{U}(N) \to \mathrm{U}(N_1)\otimes\mathrm{U}(N_2)$, $N_1 + N_2 = N$. In the field theory language this corresponds to the complex adjoint scalar in the $\mathcal{N} = 2$ vector supermultiplet having an appropriate vacuum expectation value. In the string theory language this corresponds to splitting the N D3-branes into two stacks of N_1 and N_2 D3-branes, and moving them apart by some distance X in the two real directions transverse to the D-branes untouched by the $\mathbb{Z}_2$ orbifold action.

The relevant part of the classical action in this case is given by:

$$\begin{aligned} S = &-\int_{{\rm D3}_1} a\,{\rm Tr}[F_1^{\mu\nu}F_{1\mu\nu}] + b\,\sigma\,{\rm Tr}[F_1^{\mu\nu}F_{1\mu\nu}] + c_1\,\sigma + d\,\epsilon^{\mu\nu\sigma\rho}\,C_{\mu\nu}\,{\rm Tr}[F_{1\sigma\rho}] - \\ &-\int_{{\rm D3}_2} a\,{\rm Tr}[F_2^{\mu\nu}F_{2\mu\nu}] + b\,\sigma\,{\rm Tr}[F_2^{\mu\nu}F_{2\mu\nu}] + c_2\,\sigma + d\,\epsilon^{\mu\nu\sigma\rho}\,C_{\mu\nu}\,{\rm Tr}[F_{2\sigma\rho}] - \\ &-\int_{{\rm D3}\times\mathbb{R}^2}\left(\frac{1}{2}\partial^\mu\sigma\;\partial_\mu\sigma + \frac{1}{12}H^{\mu\nu\sigma}H_{\mu\nu\sigma}\right). \end{aligned} \tag{5.10}$$

The couplings a, b, d are the same as before, while the relevant combinations containing the couplings $c_{1,2}$ are now given by

$$bc_{1,2} = 2N_{1,2}d^2\,. \tag{5.11}$$

In the following we will assume that the ${\rm D3}_{1,2}$-branes (that is, the stacks of $N_{1,2}$ D3-branes) are located at $x^i = x^i_{1,2}$, respectively, in the extra two dimensions. Here x^i, $i = 1,2$, are real coordinates, and in the following we will use the notation $X^i \equiv x^i_1 - x^i_2$, $X^2 \equiv X^iX^i$.

[22] Here we should point out that the U(1) gauge coupling can in general receive finite (string) threshold corrections. Here, however, we ignore such corrections as we are interested in the field theory limit.

Upon integrating out the bulk fields, we obtain:

$$\begin{aligned}
S_{\rm brane}^{\rm qu} = -\int \frac{d^4p}{(2\pi)^4} \Bigg(\bigg[a - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2}\, e^{ik\cdot X}\bigg] &\operatorname{Tr}\left[F_1^{\mu\nu}(p)\ F_{1\mu\nu}(-p)\right] + \\
+ \bigg[a - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2}\, e^{ik\cdot X}\bigg] &\operatorname{Tr}\left[F_2^{\mu\nu}(p)\ F_{2\mu\nu}(-p)\right] + \\
+ \frac{d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\Big(\operatorname{Tr}\left[F_1^{\mu\nu}(p)\right]\operatorname{Tr}\left[F_{1\mu\nu}(-p)\right] &+ \\
+ \operatorname{Tr}\left[F_2^{\mu\nu}(p)\right]\operatorname{Tr}\left[F_{2\mu\nu}(-p)\right]\Big) &+ \\
+ \frac{2d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\, e^{ik\cdot X}\ \operatorname{Tr}\left[F_1^{\mu\nu}(p)\right]\operatorname{Tr}\left[F_{2\mu\nu}(-p)\right]\Bigg) &. \qquad (5.12)
\end{aligned}$$

As in the previous subsection, let us separate the U(1) contributions:

$$\begin{aligned}
S_{\rm brane}^{\rm qu} = -\int \frac{d^4p}{(2\pi)^4} \Bigg(\bigg[a - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2}\, e^{ik\cdot X}\bigg] &\operatorname{Tr}\left[\widehat{F}_1^{\mu\nu}(p)\widehat{F}_{1\mu\nu}(-p)\right] + \\
+ \bigg[a - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2} e^{ik\cdot X}\bigg] &\operatorname{Tr}\left[\widehat{F}_2^{\mu\nu}(p)\widehat{F}_{2\mu\nu}(-p)\right] + \\
+ \bigg[a - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2} e^{ik\cdot X} &+ \\
+ \frac{N_1 d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\bigg] &\operatorname{Tr}\left[f_1^{\mu\nu}(p)\ f_{1\mu\nu}(-p)\right] + \\
+ \bigg[a - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2}\ e^{ik\cdot X} &+ \\
+ \frac{N_2 d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\bigg] &\operatorname{Tr}\left[f_2^{\mu\nu}(p)\ f_{2\mu\nu}(-p)\right] + \\
+ \frac{2d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\, e^{ik\cdot X}\ \operatorname{Tr}\left[f_1^{\mu\nu}(p)\right]\operatorname{Tr}\left[f_{2\mu\nu}(-p)\right]\Bigg) &. \qquad (5.13)
\end{aligned}$$

To further simplify this expression, let us explicitly take the traces over the U(1) generators. Thus, we have

$$f_1^{\mu\nu} = \frac{1}{\sqrt{2N_1}}\overline{f}_1^{\mu\nu}\operatorname{diag}(\mathbb{1}_{N_1}, O_{N_2})\,, \qquad (5.14)$$

$$f_2^{\mu\nu} = \frac{1}{\sqrt{2N_2}}\overline{f}_2^{\mu\nu}\operatorname{diag}(O_{N_1}, \mathbb{1}_{N_2})\,, \qquad (5.15)$$

where O_m is a null $m\times m$ matrix, and $\overline{f}_{1,2}^{\mu\nu}$ are the U(1) field strengths. We therefore obtain:

$$\begin{aligned}
S_{\rm brane}^{\rm qu} = -\int \frac{d^4p}{(2\pi)^4} \Bigg(\bigg[a - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2} e^{ik\cdot X}\bigg] &\operatorname{Tr}\left[\widehat{F}_1^{\mu\nu}(p)\widehat{F}_{1\mu\nu}(-p)\right] + \\
+ \bigg[a - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2} e^{ik\cdot X}\bigg] &\operatorname{Tr}\left[\widehat{F}_2^{\mu\nu}(p)\widehat{F}_{2\mu\nu}(-p)\right] +
\end{aligned}$$

$$
\begin{aligned}
&+\left[a-\frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2}-\frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2}e^{ik\cdot X}+\right.\\
&\qquad\left.+\frac{N_1d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\right]\frac{1}{2}\overline{f}_1^{\mu\nu}(p)\overline{f}_{1\mu\nu}(-p)+\\
&+\left[a-\frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2}-\frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2}e^{ik\cdot X}+\right.\\
&\qquad\left.+\frac{N_2d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\right]\frac{1}{2}\overline{f}_2^{\mu\nu}(p)\overline{f}_{2\mu\nu}(-p)+\\
&\left.+\frac{\sqrt{N_1N_2}d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}e^{ik\cdot X}\overline{f}_1^{\mu\nu}(p)\overline{f}_{2\mu\nu}(-p)\right). \qquad (5.16)
\end{aligned}
$$

Here it is convenient to rotate the above U(1)'s to the following basis

$$
\begin{aligned}
\overline{f}_+^{\mu\nu} &\equiv \cos(\alpha)\overline{f}_1^{\mu\nu}+\sin(\alpha)\overline{f}_2^{\mu\nu}\,,\\
\overline{f}_-^{\mu\nu} &\equiv -\sin(\alpha)\overline{f}_1^{\mu\nu}+\cos(\alpha)\overline{f}_2^{\mu\nu}\,,
\end{aligned} \qquad (5.17)
$$

where

$$
\cos(\alpha)=\sqrt{\frac{N_1}{N}}\,,\qquad \sin(\alpha)=\sqrt{\frac{N_2}{N}}\,. \qquad (5.18)
$$

Here $\overline{f}_+^{\mu\nu}$ is the field strength of the overall center-off-mass U(1) in $\mathrm{U}(N)=\mathrm{SU}(N)\otimes\mathrm{U}(1)_+$, while $\overline{f}_-^{\mu\nu}$ is the field strength of the U(1) in the breaking $\mathrm{SU}(N)\to\mathrm{SU}(N_1)\otimes\mathrm{SU}(N_2)\otimes\mathrm{U}(1)_-$. Indeed, we have

$$
f_1^{\mu\nu}+f_2^{\mu\nu}=\frac{1}{\sqrt{2N}}\overline{f}_+^{\mu\nu}\mathrm{diag}\left(\mathbb{1}_{N_1},\mathbb{1}_{N_2}\right)+\frac{1}{\sqrt{2N}}\overline{f}_-^{\mu\nu}\mathrm{diag}\left(-\sqrt{\frac{N_2}{N_1}}\mathbb{1}_{N_1},\sqrt{\frac{N_1}{N_2}}\mathbb{1}_{N_2}\right). \qquad (5.19)
$$

The matrices multiplying $\overline{f}_+^{\mu\nu}$ and $\overline{f}_-^{\mu\nu}$ on the r.h.s. of this equation are nothing but the properly normalized generators of $\mathrm{U}(1)_+$ and $\mathrm{U}(1)_-$, respectively.

In the new basis we have

$$
\begin{aligned}
&\int\frac{d^4p}{(2\pi)^4}\left(\left[a-\frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2}-\frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2}e^{ik\cdot X}+\frac{N_1d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\right]\frac{1}{2}\overline{f}_1^{\mu\nu}(p)\overline{f}_{1\mu\nu}(-p)+\right.\\
&\qquad\left[a-\frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2}-\frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2}e^{ik\cdot X}+\frac{N_2d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\right]\frac{1}{2}\overline{f}_2^{\mu\nu}(p)\overline{f}_{2\mu\nu}(-p)+\\
&\qquad\left.\frac{\sqrt{N_1N_2}d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}e^{ik\cdot X}\overline{f}_1^{\mu\nu}(p)\overline{f}_{2\mu\nu}(-p)\right)=\\
&=\int\frac{d^4p}{(2\pi)^4}\left(\frac{\sqrt{N_1N_2}d^2}{\pi^2}\frac{N_1-N_2}{N}\left[\int\frac{d^2k}{k^2+p^2}e^{ik\cdot X}-\int\frac{d^2k}{k^2}e^{ik\cdot X}\right]\overline{f}_+^{\mu\nu}(p)\overline{f}_{-\mu\nu}(-p)+\right.\\
&\qquad+\left(a+\frac{2N_1N_2}{N}\frac{d^2}{\pi^2}\left[\int\frac{d^2k}{k^2+p^2}e^{ik\cdot X}-\int\frac{d^2k}{k^2}e^{ik\cdot X}\right]\right)\frac{1}{2}\overline{f}_+^{\mu\nu}(p)\overline{f}_{+\mu\nu}(-p)+\\
&\qquad+\left(a-\frac{d^2}{N\pi^2}\left[2N_1N_2\int\frac{d^2k}{k^2+p^2}e^{ik\cdot X}+\right.\right.\\
&\qquad\qquad\left.\left.\left.+\left[N_1^2+N_2^2\right]\int\frac{d^2k}{k^2}e^{ik\cdot X}\right]\right)\frac{1}{2}\overline{f}_-^{\mu\nu}(p)\overline{f}_{-\mu\nu}(-p)\right). \qquad (5.20)
\end{aligned}
$$

In arriving at this equation we have used (5.11), as well as the fact that, as we discussed in the previous subsection, in the integral

$$\int \frac{d^2k}{k^2} \tag{5.21}$$

the IR cut-off is given by $k^2 = p^2$, so that

$$\frac{bc_{1,2}}{2\pi^2}\int_{k^2=p^2}\frac{d^2k}{k^2} - \frac{N_{1,2}d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2} = 0\,. \tag{5.22}$$

In fact, the r.h.s. of (5.20) further simplifies once we go to the field theory limit. This limit is given by

$$\alpha' \to 0\,, \qquad X^i \to 0\,, \qquad \phi^i \equiv \frac{X^i}{\alpha'} = \text{fixed}\,. \tag{5.23}$$

In this limit the two stacks of D3-branes come on top of each other, but the gauge symmetry is still $\mathrm{U}(N_1)\otimes\mathrm{U}(N_2)$ — the original $\mathrm{U}(N)$ gauge group is broken by the adjoint Higgs vacuum expectation value parametrized by ϕ^i. In particular, in this limit we have

$$\int\frac{d^2k}{k^2+p^2}e^{ik\cdot X} - \int\frac{d^2k}{k^2}e^{ik\cdot X} \to 0\,. \tag{5.24}$$

This implies that on the r.h.s. of (5.20) the $\overline{f}_+^{\mu\nu}\overline{f}_{-\mu\nu}$ term goes to zero, the coupling for the $\overline{f}_+^{\mu\nu}\overline{f}_{+\mu\nu}$ term goes to $a/2$ (that is, this coupling is not renormalized, which is consistent with the fact that the overall center-of-mass U(1) should not run), while the coupling for the $\overline{f}_-^{\mu\nu}\overline{f}_{-\mu\nu}$ term is renormalized as follows:

$$\widetilde{a}_-(p^2,\phi^2) = a - \frac{Nd^2}{\pi^2}\int\frac{d^2k}{k^2}e^{ik\cdot X}\,. \tag{5.25}$$

This coupling, as well as the non-abelian gauge couplings, need to be regularized. This regularization, however, depends on whether $p^2 \gg \phi^2$ or $p^2 \ll \phi^2$. This is because the aforementioned $\alpha' \to 0$ limit must be taken differently depending on p^2. However, (5.24) holds regardless of how the limit is taken.

Collecting the above results we see that the effective quantum action is given by:

$$\begin{aligned}
S^{\rm qu}_{\rm brane} = -\int\frac{d^4p}{(2\pi)^4}\Big(&\widetilde{a}_1(p^2,\phi^2)\,\mathrm{Tr}\left[\widehat{F}_1^{\mu\nu}(p)\widehat{F}_{1\mu\nu}(-p)\right] + \\
&+\widetilde{a}_2(p^2,\phi^2)\,\mathrm{Tr}\left[\widehat{F}_2^{\mu\nu}(p)\widehat{F}_{2\mu\nu}(-p)\right] + \\
&+\frac{1}{2}a\overline{f}_+^{\mu\nu}(p)\overline{f}_{+\mu\nu}(-p) + \frac{1}{2}\widetilde{a}_-(p^2,\phi^2)\overline{f}_-^{\mu\nu}(p)\overline{f}_{-\mu\nu}(-p)\Big)\,,
\end{aligned} \tag{5.26}$$

where the $\mathrm{U}(1)_-$ coupling is given by (5.25), while the non-abelian couplings are given by:

$$\begin{aligned}
\widetilde{a}_1(p^2,\phi^2) &= a - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2}e^{ik\cdot X}\,,\\
\widetilde{a}_2(p^2,\phi^2) &= a - \frac{bc_2}{2\pi^2}\int\frac{d^2k}{k^2} - \frac{bc_1}{2\pi^2}\int\frac{d^2k}{k^2}e^{ik\cdot X}\,.
\end{aligned} \tag{5.27}$$

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As before, the integral (5.21) is regularized as follows:

$$\int_{k^2=p^2}^{k^2=\Lambda^2} \frac{d^2k}{k^2} = \pi \ln\left(\frac{\Lambda^2}{p^2}\right). \tag{5.28}$$

However, as we have already mentioned, the regularization of the integral

$$\int \frac{d^2k}{k^2} e^{ik\cdot X} \tag{5.29}$$

depends upon p^2. For $p^2 \gg \phi^2$ we take the $\alpha' \to 0$ limit in the exponent

$$\int \frac{d^2k}{k^2} e^{ik\cdot X} \to \int \frac{d^2k}{k^2}, \tag{5.30}$$

and regularize the resulting integral as in (5.28). For $p^2 \gg \phi^2$ we therefore have

$$2\widetilde{a}_1 = 2\widetilde{a}_2 = 2\widetilde{a}_- = \frac{1}{g^2_{\mathrm{YM}}} + \frac{\beta_0}{16\pi^2} \ln\left(\frac{\Lambda^2}{p^2}\right). \tag{5.31}$$

The r.h.s. of this equation is nothing but the one-loop renormalized $\mathrm{SU}(N)$ gauge coupling with the β-function coefficient $\beta_0 = -2N$. All three gauge couplings, that is, those for $\mathrm{SU}(N_1)$, $\mathrm{SU}(N_2)$ and $\mathrm{U}(1)_-$, run together at $p^2 \gg \phi^2$ as at large momenta the effects of the $\mathrm{SU}(N) \to \mathrm{SU}(N_1) \otimes \mathrm{SU}(N_2) \otimes \mathrm{U}(1)_-$ breaking are negligible.

Now, at small momenta $p^2 \ll \phi^2$ the $\alpha' \to 0$ limit must be taken differently. In particular, in the integral (5.29) we first redefine the integration variables via $\zeta^i \equiv \alpha' k^i$. The resulting integral

$$\int \frac{d^2\zeta}{\zeta^2} e^{i\zeta\cdot\phi} \tag{5.32}$$

must then be regularized for small ζ^2. Note that ζ^2 has dimension of $(\text{length})^2$, so this is a UV divergence. The regularized integral (5.32) is then given by

$$\pi \ln\left(\frac{\xi^2\Lambda^2}{\phi^2}\right), \tag{5.33}$$

where ξ parametrizes the subtraction scheme dependence (see below). For small momenta $p^2 \ll \phi^2$ we therefore have:

$$\begin{aligned}
2\widetilde{a}_1 &= \frac{1}{g^2_{\mathrm{YM}}} + \frac{\beta_0^{(1)}}{16\pi^2} \ln\left(\frac{\Lambda^2}{p^2}\right) + \frac{\beta_0^{(2)}}{16\pi^2} \ln\left(\frac{\xi^2\Lambda^2}{\phi^2}\right), \\
2\widetilde{a}_2 &= \frac{1}{g^2_{\mathrm{YM}}} + \frac{\beta_0^{(2)}}{16\pi^2} \ln\left(\frac{\Lambda^2}{p^2}\right) + \frac{\beta_0^{(1)}}{16\pi^2} \ln\left(\frac{\xi^2\Lambda^2}{\phi^2}\right), \\
2\widetilde{a}_- &= \frac{1}{g^2_{\mathrm{YM}}} + \frac{\beta_0}{16\pi^2} \ln\left(\frac{\xi^2\Lambda^2}{\phi^2}\right),
\end{aligned} \tag{5.34}$$

where $\beta_0^{(1)} = -2N_1$ and $\beta_0^{(2)} = -2N_2$ are the one-loop beta function coefficients for $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$, respectively. In the above expressions the terms proportional to $\ln(\xi^2\Lambda^2/\phi^2)$ correspond to the *threshold* corrections due to the massive gauge bosons (that is, the gauge bosons that become heavy in the Higgs mechanism).

As usual, to connect the gauge coupling evolution above and below the threshold, we need to specify a subtraction scheme. Thus, we can choose the subtraction scheme where the gauge couplings are matched at the scale $p^2 = M^2$, where M is the mass of the heavy gauge bosons. In particular, this implies that

$$\xi^2 = \frac{\phi^2}{M^2}\,. \tag{5.35}$$

We then have the following gauge coupling running. For $p^2 \geq M^2$

$$2\widetilde{a}_1 = 2\widetilde{a}_2 = 2\widetilde{a}_- = \frac{1}{\widetilde{g}^2_{\mathrm{YM}}(p^2)}\,, \tag{5.36}$$

where

$$\frac{1}{\widetilde{g}^2_{\mathrm{YM}}(p^2)} \equiv \frac{1}{g^2_{\mathrm{YM}}} + \frac{\beta_0}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right), \tag{5.37}$$

which is the $\mathrm{SU}(N)$ gauge coupling at $p^2 \geq M^2$. For $p^2 < M^2$ we have

$$\begin{aligned}
2\widetilde{a}_1 &= \frac{1}{\widetilde{g}^2_{\mathrm{YM}}(M^2)} + \frac{\beta_0^{(1)}}{16\pi^2}\ln\left(\frac{M^2}{p^2}\right),\\
2\widetilde{a}_2 &= \frac{1}{\widetilde{g}^2_{\mathrm{YM}}(M^2)} + \frac{\beta_0^{(2)}}{16\pi^2}\ln\left(\frac{M^2}{p^2}\right),\\
2\widetilde{a}_- &= \frac{1}{\widetilde{g}^2_{\mathrm{YM}}(M^2)}\,,
\end{aligned} \tag{5.38}$$

so that below the threshold the $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$ gauge couplings run with the β-function coefficients $\beta_0^{(1)} = -2N_1$ and $\beta_0^{(2)} = -2N_2$, respectively, while the $\mathrm{U}(1)_-$ gauge coupling does not run at all.

Thus, using the brane-bulk duality approach we reproduce the expected perturbative running of gauge couplings in the corresponding $\mathcal{N}=2$ gauge theories. In the brane-bulk duality approach, however, we do not perform any loop computations. Rather, the information about the loop corrections in gauge theory is encoded in the corresponding classical higher dimensional field theory. This is a consequence of the fact that the brane-bulk duality simply follows from the closed-open string duality.

Before we end this subsection, the following remark is in order. In the above computations we had to regularize various (logarithmically) divergent integrals. The corresponding regularizations depend upon the four-dimensional momentum squared. For instance, in this subsection we saw that the regularization of the integrals containing the information about the threshold depends on whether p^2 is above or below the threshold. In particular, the details of the corresponding regularization are somewhat different from what happens in the direct loop computation in the field theory language. This appears to be a common feature of string theory embeddings of gauge theories as far as computations of, say, gauge coupling running are concerned. In particular, to reproduce the gauge coupling running, it appears to be necessary to introduce an (IR) cut-off which is p^2 dependent. This appears to be a consequence of the fact that such computations are typically done within the *on-shell* formulation of string theory.[23]

5.3 Cases with matter

For completeness in this subsection we would like to briefly discuss examples of $\mathcal{N}=2$ gauge theories with matter. The simplest examples of this type are obtained via the aforementioned $\mathbb{Z}_2$ orbifold construction with the twisted Chan-Paton matrix given by $\gamma_R = \mathrm{diag}(\mathbb{1}_{N_1}, -\mathbb{1}_{N_2})$, where $N_1 + N_2 = N$. In this case the gauge group is $\mathrm{U}(N_1)\otimes \mathrm{U}(N_2)$, and we have matter hypermultiplets in $(\mathbf{N}_1, \overline{\mathbf{N}}_2)(+1,-1)$ and $(\overline{\mathbf{N}}_1, \mathbf{N}_2)(-1,+1)$, where the U(1) charges are given in parentheses. Note that all N D3-branes are now coincident, and the gauge symmetry is broken due to the non-trivial orbifold action on the Chan-Paton factors.

The relevant part of the classical action is given by (for simplicity the matter field contributions are not shown, and we use the notation $F^{\mu\nu} = \mathrm{diag}(F_1^{\mu\nu}, F_2^{\mu\nu})$):

$$\begin{aligned} S =& -\int_{\mathrm{D3}} \Big(a\,\mathrm{Tr}\,[F^{\mu\nu}F_{\mu\nu}] + b\sigma\,\mathrm{Tr}\,[\gamma_R F^{\mu\nu}F_{\mu\nu}] + \mathrm{Tr}(\gamma_R)\widehat{c}\sigma + d\epsilon^{\mu\nu\sigma\rho}C_{\mu\nu}\,\mathrm{Tr}\,[\gamma_R F_{\sigma\rho}]\Big) - \\ & - \int_{\mathrm{D3}\times\mathbb{R}^2}\left(\frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \frac{1}{12}H^{\mu\nu\sigma}H_{\mu\nu\sigma}\right) \\ =& -\int_{\mathrm{D3}}\Big(a\Big[\mathrm{Tr}\Big[F_1^{\mu\nu}F_{1\mu\nu}\Big] + \mathrm{Tr}\,[F_2^{\mu\nu}F_{2\mu\nu}]\Big] + b\sigma\Big[\mathrm{Tr}\,[F_1^{\mu\nu}F_{1\mu\nu}] - \mathrm{Tr}\,[F_2^{\mu\nu}F_{2\mu\nu}]\Big] + \\ & \qquad + (N_1 - N_2)\widehat{c}\sigma + d\epsilon^{\mu\nu\sigma\rho}C_{\mu\nu}\Big[\mathrm{Tr}\,[F_{1\sigma\rho}] - \mathrm{Tr}\,[F_{2\sigma\rho}]\Big]\Big) - \\ & - \int_{\mathrm{D3}\times\mathbb{R}^2}\left(\frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \frac{1}{12}H^{\mu\nu\sigma}H_{\mu\nu\sigma}\right), \end{aligned} \tag{5.39}$$

where the couplings a, b, d are the same as before, while

$$b\widehat{c} = 2d^2\,. \tag{5.40}$$

Note that for $N_1 = 0$ or $N_2 = 0$ we recover the action (5.1).

Integrating out the bulk fields, we obtain:

$$S^{\mathrm{qu}}_{\mathrm{brane}} = -\int \frac{d^4p}{(2\pi)^4}\left(\left[a - \frac{b\widehat{c}}{2\pi^2}(N_1 - N_2)\int\frac{d^2k}{k^2}\right]\mathrm{Tr}\,[F_1^{\mu\nu}(p)F_{1\mu\nu}(-p)] + \right.$$

[23] We would like to thank Tom Taylor for a discussion on this issue.

$$
\begin{aligned}
&+\left[a-\frac{b\widehat{c}}{2\pi^2}(N_2-N_1)\int\frac{d^2k}{k^2}\right]\mathrm{Tr}\left[F_2^{\mu\nu}(p)F_{2\mu\nu}(-p)\right]+\\
&+\frac{d^2}{\pi^2}\int\frac{d^2k}{k^2+p^2}\left(\mathrm{Tr}[F_1^{\mu\nu}(p)]-\mathrm{Tr}[F_2^{\mu\nu}(p)]\right)\times\\
&\qquad\times\left(\mathrm{Tr}[F_{1\mu\nu}(-p)]-\mathrm{Tr}[F_{2\mu\nu}(-p)]\right)\Bigg). \qquad (5.41)
\end{aligned}
$$

The U(1) contributions can be extracted as in the previous subsection, and, in fact, all the corresponding normalizations are exactly the same as before. We therefore obtain:

$$
\begin{aligned}
S_{\mathrm{brane}}^{\mathrm{qu}}=-\int\frac{d^4p}{(2\pi)^4}\Bigg(&\widetilde{a}_1(p^2)\,\mathrm{Tr}\left[\widehat{F}_1^{\mu\nu}(p)\widehat{F}_{1\mu\nu}(-p)\right]+\widetilde{a}_2(p^2)\,\mathrm{Tr}\left[\widehat{F}_2^{\mu\nu}(p)\widehat{F}_{2\mu\nu}(-p)\right]+\\
&+\frac{1}{2}a\overline{f}_+^{\mu\nu}(p)\overline{f}_{+\mu\nu}(-p)+\frac{1}{2}\widetilde{a}_-(p^2)\overline{f}_-^{\mu\nu}(p)\overline{f}_{-\mu\nu}(-p)\Bigg), \qquad (5.42)
\end{aligned}
$$

where the $\mathrm{U}(1)_-$ coupling is given by

$$
2\widetilde{a}_-=\frac{1}{g_{\mathrm{YM}}^2}+\frac{\beta_0^{(-)}}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right), \qquad (5.43)
$$

while the non-abelian couplings are given by

$$
2\widetilde{a}_{1,2}=\frac{1}{g_{\mathrm{YM}}^2}+\frac{\beta_0^{(1,2)}}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right), \qquad (5.44)
$$

where $\beta_0^{(-)}=2N$ is the $\mathrm{U}(1)_-$ one-loop β-function coefficient, while $\beta_0^{(1)}=-2(N_1-N_2)$ and $\beta_0^{(2)}=-2(N_2-N_1)$ are the one-loop β-function coefficients for the $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$ subgroups, respectively. Thus, as we see, once again, we correctly reproduce the running of the gauge couplings for the $\mathrm{SU}(N_1)\otimes\mathrm{SU}(N_2)\otimes\mathrm{U}(1)_-$ subgroup (the overall center-of-mass $\mathrm{U}(1)_+$ does not run as there is no matter charged under it).

Note that the twisted tadpole for σ vanishes for $N_1=N_2$, that is, for $\mathrm{Tr}(\gamma_R)=0$. The non-abelian one-loop β-function coefficients in this case vanish. The $\mathrm{U}(1)_-$ one-loop β-function coefficient, however, is still non-vanishing, so the $\mathrm{U}(1)_-$ gauge coupling runs. In the large-N limit $\mathrm{U}(1)_-$ decouples in the IR, and we are left with an $\mathcal{N}=2$ superconformal field theory.

6. Examples with $\mathcal{N}=1$ supersymmetry

In this section we would like to discuss the brane-bulk duality in $\mathcal{N}=1$ supersymmetric theories of the type discussed in subsection 3.1. Thus, let us consider the example where $\Gamma=\mathbb{Z}_2\otimes\mathbb{Z}_2$, and the action of the generators R_1 and R_2 of the two $\mathbb{Z}_2$'s on the complex coordinates z_α on $\mathcal{M}=\mathcal{C}^3/\Gamma$ is given by (here $R_3\equiv R_1R_2$)

$$
R_\alpha:-z_\beta\to(-1)^{\delta_{\alpha\beta}}z_\beta\,. \qquad (6.1)
$$

Let us choose the twisted Chan-Paton matrices as follows: $\gamma_{R_1} = \mathrm{diag}(\mathbb{1}_{N_1}, \mathbb{1}_{N_2})$, $\gamma_{R_2} = \gamma_{R_3} = \mathrm{diag}(\mathbb{1}_{N_1}, -\mathbb{1}_{N_2})$, where $N_1 + N_2 = N$. In this case we have the $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N_1) \otimes \mathrm{U}(N_2)$ gauge theory with chiral matter in $(\mathbf{N}_1, \overline{\mathbf{N}}_2)(+1, -1)$ and $(\overline{\mathbf{N}}_1, \mathbf{N}_2)(-1, +1)$, where the U(1) charges are given in parentheses. Note that all N D3-branes are coincident, and the gauge symmetry is broken due to the non-trivial orbifold action on the Chan-Paton factors. For $N_1 = N_2$ we have an $\mathcal{N} = 1$ theory of the type discussed in subsection 3.1. In particular, in the planar limit the correlation functions in this theory are the same as in the parent $\mathcal{N} = 2$ supersymmetric gauge theory with $\mathrm{U}(N)$ gauge group and no matter. We can therefore discuss this theory in the context of the brane-bulk duality as in the large-N limit we have the corresponding non-renormalization theorem beyond the one-loop order.

For calculational convenience in the following we will keep N_1 and N_2 arbitrary. The calculation of the one-loop effective quantum action then gives a correct result even for $N_1 \neq N_2$, but only for $N_1 = N_2$ do we have the non-renormalization theorem beyond the one-loop order. The relevant part of the classical action is given by (for simplicity the matter field contributions are not shown, and we use the notation $F^{\mu\nu} = \mathrm{diag}(F_1^{\mu\nu}, F_2^{\mu\nu})$):

$$
\begin{aligned}
S = & -\int_{\mathrm{D3}} \bigg(a \,\mathrm{Tr}\left[F^{\mu\nu} F_{\mu\nu}\right] + \frac{1}{\sqrt{2}} \sum_\alpha \Big[b \sigma_\alpha \,\mathrm{Tr}\left[\gamma_{R_\alpha} F^{\mu\nu} F_{\mu\nu}\right] + \mathrm{Tr}(\gamma_{R_\alpha}) \widehat{c} \sigma_\alpha + \\
& \qquad + d \epsilon^{\mu\nu\sigma\rho} C_{\alpha\mu\nu} \,\mathrm{Tr}\left[\gamma_{R_\alpha} F_{\sigma\rho}\right] \Big] \bigg) - \\
& - \sum_\alpha \int_{\mathrm{D3} \times \mathcal{F}_\alpha} \left(\frac{1}{2} \partial^\mu \sigma_\alpha \partial_\mu \sigma_\alpha + \frac{1}{12} H_\alpha^{\mu\nu\sigma} H_{\alpha\mu\nu\sigma} \right), \qquad (6.2)
\end{aligned}
$$

where the couplings $a, b, \widehat{c}, d$ are the same as before, and the overall factor of $1/\sqrt{2}$ is due to the fact that the R_α twisted fields σ_α and $C_\alpha^{\mu\nu}$ now propagate in $\mathbb{R}^{1,3} \times \mathcal{F}_\alpha$, where each fixed point set $\mathcal{F}_\alpha$ is an orbifold $\mathbb{R}^2/\mathbb{Z}_2$.

Integrating out the bulk fields, we obtain:

$$
\begin{aligned}
S_{\mathrm{brane}}^{\mathrm{qu}} = & -\int \frac{d^4 p}{(2\pi)^4} \Bigg(\left[a - \frac{b\widehat{c}}{4\pi^2} (3N_1 - N_2) \int \frac{d^2 k}{k^2} \right] \mathrm{Tr}\left[F_1^{\mu\nu}(p) F_{1\mu\nu}(-p)\right] + \\
& + \left[a - \frac{b\widehat{c}}{4\pi^2} (3N_2 - N_1) \int \frac{d^2 k}{k^2} \right] \mathrm{Tr}\left[F_2^{\mu\nu}(p) F_{2\mu\nu}(-p)\right] + \\
& + \frac{d^2}{2\pi^2} \int \frac{d^2 k}{k^2 + p^2} \left(\mathrm{Tr}[F_1^{\mu\nu}(p)] + \mathrm{Tr}[F_2^{\mu\nu}(p)] \right) \times \\
& \qquad \times \left(\mathrm{Tr}[F_{1\mu\nu}(-p)] + \mathrm{Tr}[F_{2\mu\nu}(-p)] \right) + \\
& + \frac{d^2}{\pi^2} \int \frac{d^2 k}{k^2 + p^2} \left(\mathrm{Tr}[F_1^{\mu\nu}(p)] - \mathrm{Tr}[F_2^{\mu\nu}(p)] \right) \times \\
& \qquad \times \left(\mathrm{Tr}[F_{1\mu\nu}(-p)] - \mathrm{Tr}[F_{2\mu\nu}(-p)] \right) \Bigg). \qquad (6.3)
\end{aligned}
$$

The U(1) contributions can be extracted as in the previous section, and, in fact, all the corresponding normalizations are exactly the same as before. We therefore obtain:

$$S_{\text{brane}}^{\text{qu}} = -\int \frac{d^4p}{(2\pi)^4}\Big(\widetilde{a}_1(p^2)\,\mathrm{Tr}\left[\widehat{F}_1^{\mu\nu}(p)\widehat{F}_{1\mu\nu}(-p)\right] + \widetilde{a}_2(p^2)\,\mathrm{Tr}\left[\widehat{F}_2^{\mu\nu}(p)\widehat{F}_{2\mu\nu}(-p)\right] + \\ + \frac{1}{2}a\overline{f}_+^{\mu\nu}(p)\overline{f}_{+\mu\nu}(-p) + \frac{1}{2}\widetilde{a}_-(p^2)\overline{f}_-^{\mu\nu}(p)\overline{f}_{-\mu\nu}(-p)\Big)\,, \tag{6.4}$$

where the $\mathrm{U}(1)_-$ coupling is given by

$$2\widetilde{a}_- = \frac{1}{g_{\mathrm{YM}}^2} + \frac{\beta_0^{(-)}}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right), \tag{6.5}$$

while the non-abelian couplings are given by

$$2\widetilde{a}_{1,2} = \frac{1}{g_{\mathrm{YM}}^2} + \frac{\beta_0^{(1,2)}}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right), \tag{6.6}$$

where $\beta_0^{(-)} = N$ is the $\mathrm{U}(1)_-$ one-loop β-function coefficient, while $\beta_0^{(1)} = -(3N_1 - N_2)$ and $\beta_0^{(2)} = -(3N_2 - N_1)$ are the one-loop β-function coefficients for the $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$ subgroups, respectively. Thus, we correctly reproduce the running of the gauge couplings for the $\mathrm{SU}(N_1)\otimes\mathrm{SU}(N_2)\otimes\mathrm{U}(1)_-$ subgroup (the overall center-of-mass $\mathrm{U}(1)_+$ does not run as there is no matter charged under it).

Note that the twisted tadpole for σ_1 does not vanish, so that the non-abelian part of the gauge theory is non-conformal even for $N_1 = N_2 \equiv M$. In the large-M limit $\mathrm{U}(1)_-$ decouples in the IR, and we are left with the $\mathcal{N}=2$ supersymmetric $\mathrm{SU}(M)\times\mathrm{SU}(M)$ gauge theory with chiral matter in $(\mathbf{M},\overline{\mathbf{M}})$ and $(\overline{\mathbf{M}},\mathbf{M})$.

7. Non-supersymmetric theories

So far we have been focusing on $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetric theories. However, in the large-N limit we can also discuss certain non-trivial non-supersymmetric cases as well. The large-N property is crucial here. The reason is that in the cases where the orbifold group $\Gamma \not\subset \mathrm{SU}(3)$, we always have twisted NS-NS closed string sectors with tachyons. Their contributions to the corresponding part of the annulus amplitude (2.8) is then exponentially divergent unless we require that

$$\mathrm{Tr}\,(\gamma_a) = 0\,, \qquad g_a \notin \mathrm{SU}(3)\,. \tag{7.1}$$

Even if this condition is satisfied, we must take the 't Hooft limit — indeed, otherwise it is unclear, for instance, how to deal with the diagrams with handles, which contain tachyonic divergences. In fact, the same applies to some non-planar diagrams without handles, that is, diagrams where the external lines are attached to more than one boundaries (such diagrams are subleading in the large-N limit).

To obtain well-defined non-supersymmetric non-conformal models, consider an orbifold group $\Gamma \subset \mathrm{Spin}(6)$, which is not a subgroup of SU(3). Let $\widetilde{\Gamma}$ be a non-trivial subgroup of Γ such that $\widetilde{\Gamma} \subset \mathrm{SU}(2)$. We will allow the Chan-Paton matrices γ_a corresponding to the twists $g_a \in \widetilde{\Gamma}$ (which have $d_a = 2$) not to be traceless, so that the corresponding $\mathcal{N} = 2$ model is not conformal. However, we will require that the other Chan-Paton matrices γ_a for the twists $g_a \notin \widetilde{\Gamma}$ be traceless. The resulting non-supersymmetric model is not conformal. However, in the planar limit the perturbative gauge theory amplitudes are not renormalized beyond one loop (as usual, various running U(1)'s decouple in the IR in this limit). The proof of this statement is completely parallel to that we gave in subsection 3.1 for $\mathcal{N} = 1$ theories.

Let us consider a simple example of such a theory. Let $\Gamma \approx \mathbb{Z}_2 \otimes \mathbb{Z}_3$, where the action of the generators R and θ of the $\mathbb{Z}_2$ respectively $\mathbb{Z}_3$ subgroups on the complex coordinates z_α on $\mathcal{M} = \mathbb{C}^3/\Gamma$ is as follows: $R : z_1 \to z_1$, $R : z_{2,3} \to -z_{2,3}$, $\theta : z_1 \to \omega z_1$, $\theta : z_{2,3} \to z_{2,3}$, where $\omega \equiv \exp(2\pi i/3)$. The twisted Chan-Paton matrices are given by: $\gamma_R = \mathbb{1}_{3N}$, $\gamma_\theta = \mathrm{diag}(\mathbb{1}_N, \omega\mathbb{1}_N, \omega^{-1}\mathbb{1}_N)$. Then the theory is a non-supersymmetric $\mathrm{U}(N) \otimes \mathrm{U}(N) \otimes \mathrm{U}(N)$ gauge theory with matter consisting of complex scalars in $(\mathbf{N}, \overline{\mathbf{N}}, \mathbf{1})$, $(\mathbf{1}, \mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{1}, \mathbf{N})$, as well as chiral fermions in the above representations plus their complex conjugates.

The gauge coupling renormalization in this model can be discussed in complete parallel with the previous sections. Since the U(1)'s decouple in the IR in the large-N limit, we will ignore them in the following.[24] Then the relevant part of the classical action is given by (for simplicity the matter field contributions are not shown, and we use the notation $F^{\mu\nu} = \mathrm{diag}(F_1^{\mu\nu}, F_2^{\mu\nu}, F_3^{\mu\nu})$):

$$
\begin{aligned}
S = &- \int_{\mathrm{D3}} \left(a\,\mathrm{Tr}\left[\widehat{F}^{\mu\nu}\widehat{F}_{\mu\nu}\right] + \frac{1}{\sqrt{3}}\left[b\sigma\,\mathrm{Tr}\left[\gamma_R \widehat{F}^{\mu\nu}\widehat{F}_{\mu\nu}\right] + \mathrm{Tr}(\gamma_R)\widehat{c}\sigma\right]\right) - \\
&- \int_{\mathrm{D3}\times\mathcal{F}} \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma\,,
\end{aligned}
\tag{7.2}
$$

where the couplings $a, b, \widehat{c}$ are the same as before, and the overall factor of $1/\sqrt{3}$ is due to the fact that the R twisted field σ now propagates in $\mathbb{R}^{1,3} \times \mathcal{F}$, where the fixed point set $\mathcal{F}$ is an orbifold $\mathbb{R}^2/\mathbb{Z}_3$.

Integrating out the bulk fields, we obtain ($r = 1, 2, 3$):

$$
S^{\mathrm{qu}}_{\mathrm{brane}} = -\int \frac{d^4p}{(2\pi)^4} \sum_r \widetilde{a}_r(p^2)\,\mathrm{Tr}\left[\widehat{F}_r^{\mu\nu}(p)\widehat{F}_{r\mu\nu}(-p)\right], \tag{7.3}
$$

[24] As we saw in the previous sections, U(1) runnings at one loop receive contributions from non-planar diagrams with the two external lines corresponding to the U(1) gauge bosons attached to two different boundaries. Note, however, that these contributions are due to R-R exchanges (and the R-R sectors do not contain tachyons), so that tachyons do not contribute to this.

where the non-abelian couplings are given by

$$2\widetilde{a}_r = \frac{1}{g_{\rm YM}^2} + \frac{\beta_0^{(r)}}{16\pi^2} \ln\left(\frac{\Lambda^2}{p^2}\right). \tag{7.4}$$

Here $\beta_0^{(r)} = -2N$ are the one-loop β-function coefficients for the SU(N) subgroups.[25]

8. Comments on non-perturbative brane-bulk duality

In the previous sections we discussed the brane-bulk duality in the context of $\mathcal{N}=2$ as well as certain $\mathcal{N}=1$ and $\mathcal{N}=0$ large-N non-conformal gauge theories. In particular, we saw that in the planar limit perturbatively (on-shell) correlators in the corresponding $\mathcal{N}=0,1$ theories are the same as in the parent $\mathcal{N}=2$ theories. Moreover, the one-loop effective quantum action in such theories, which in the large-N limit is not perturbatively renormalized beyond one-loop, can be computed by performing a classical computation in a higher dimensional field theory. In this section we would like to discuss whether the non-perturbative corrections modify the brane-bulk duality picture in such theories.

One of the key simplifying features here is the large-N limit. Thus, let us consider the $\mathcal{N}=2$ supersymmetric SU(N) gauge theory without matter. In this theory the low energy effective action can be described in terms of a prepotential $\mathcal{F}$, which perturbatively does not receive corrections beyond one loop. The non-perturbative corrections come from instantons [20]:

$$\mathcal{F}_{\text{non-pert}} = \sum_{k=1}^{\infty} \mathcal{F}_k \Lambda_*^{2Nk}\,, \tag{8.1}$$

where Λ_* is the dynamically generated scale of the theory:

$$\Lambda_* = \mu \exp\left(\frac{8\pi^2}{g_{\rm YM}^2(\mu)\beta_0}\right). \tag{8.2}$$

Here $g_{\rm YM}(\mu)$ is the Yang-Mills gauge coupling at some high scale μ, and $\beta_0 = -2N$ is the one-loop β-function coefficient. Thus, the instanton corrections are weighted with

$$\Lambda_*^{2Nk} = \mu^{2Nk} \exp\left(-\frac{8\pi^2 Nk}{\lambda(\mu)}\right), \tag{8.3}$$

where $\lambda(\mu) \equiv g_{\rm YM}^2(\mu)N$ is the effective 't Hooft coupling. Note that these weights go to zero in the 't Hooft limit, which implies that the low energy effective action is not renormalized beyond one loop in the large-N limit.

[25] Here we should point out that, if the parent $\mathcal{N}=2$ theory has no matter hypermultiplets, regardless of whether the final model has $\mathcal{N}=0$ or $\mathcal{N}=1$ supersymmetry, in the above construction the one-loop β-function coefficients for the non-abelian subgroups are always given by $\widetilde{\beta}_0/|\Gamma|$, where $\widetilde{\beta}_0$ is the one-loop β-function coefficient of the non-abelian subgroup in the parent $\mathcal{N}=2$ theory.

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Next consider the $\mathcal{N}=1$ and $\mathcal{N}=0$ orbifold theories discussed in subsection 3.1 (as well as section 6) and section 7, respectively. Due to their underlying $\mathcal{N}=2$ structure, in these theories we might hope that the non-perturbative corrections to the low energy effective action also vanish in the large-N limit. If so, then we have non-trivial statements about infinitely many non-trivial $\mathcal{N}=0,1$ gauge theories, in particular, that in such theories the low energy effective action is not renormalized beyond one loop in the planar limit. Checking this conjecture in the non-supersymmetric case is rather non-trivial, but in $\mathcal{N}=1$ cases we can perform some partial checks. In particular, this conjecture implies that in the large-N limit the superpotential should not receive non-perturbative corrections, so that the classical superpotential should be exact as the superpotential does not receive any loop corrections in $\mathcal{N}=1$ supersymmetric theories. This statement can indeed be checked explicitly for such theories. Instead of being most general, here we will consider the simplest example of such an $\mathcal{N}=1$ theory (other $\mathcal{N}=1$ cases can be discussed in a similar fashion). Thus, consider the example discussed in section 6. In this example we have $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)\otimes\mathrm{SU}(N)$ gauge theory with chiral matter supermultiplets in $(\mathbf{N},\overline{\mathbf{N}})$, and $(\overline{\mathbf{N}},\mathbf{N})$. To simplify the discussion, let us take the gauge coupling of the second $\mathrm{SU}(N)$ factor to be much smaller than that of the first one. Then the second $\mathrm{SU}(N)$ can be treated as the global symmetry group for the first $\mathrm{SU}(N)$, and we have the $\mathrm{SU}(N)$ gauge theory with N flavors of quarks $Q^i,\overline{Q}_{\bar{j}}$, $i,\bar{j}=1,\ldots,N$, where Q^i and $\overline{Q}_{\bar{j}}$ transform in the fundamental $\mathbf{N}$ respectively anti-fundamental $\overline{\mathbf{N}}$ of the gauge group $\mathrm{SU}(N)$. The low energy dynamics is described in terms of the gauge invariant degrees of freedom given by the mesons $M^i_{\bar{j}}$ and baryons $B,\overline{B}$ [21]:

$$\begin{aligned} M^i_{\bar{j}} &\equiv Q^i\overline{Q}_{\bar{j}}\,, \\ B &\equiv \epsilon_{i_1,\ldots,i_N}Q^{i_1}\cdots Q^{i_N}\,, \\ \overline{B} &\equiv \epsilon^{\bar{j}_1\ldots\bar{j}_N}\overline{Q}_{\bar{j}_1}\cdots\overline{Q}_{\bar{j}_N}\,. \end{aligned} \tag{8.4}$$

The classical moduli space in this theory receives quantum corrections, which can be accounted for via the following superpotential (here A is a Lagrange multiplier related to the "glueball" field via $A\Lambda_*^{2N}=W^\alpha W_\alpha$) [21]:

$$\mathcal{W}_{\text{non-pert}}=A\left(\det(M)-B\overline{B}-\Lambda_*^{2N}\right), \tag{8.5}$$

where Λ_* is the dynamically generated scale of the theory, which is given by:

$$\Lambda_*=\mu\exp\left(-\frac{4\pi^2}{\lambda(\mu)}\right). \tag{8.6}$$

Once again, Λ_*^{2N} goes to zero in the large-N limit, so that the classical constraint

$$\det(M)-B\overline{B}=0 \tag{8.7}$$

is unmodified in this limit. Thus, in this theory the classical superpotential, which vanishes, is indeed exact in the large-N limit.[26]

The above discussion suggests that the brane-bulk duality discussed in the previous sections in the context of the aforementioned gauge theories might hold even non-perturbatively, so that the corresponding low energy effective quantum action is not renormalized beyond one loop in the large-N limit.

9. Concluding remarks

We would like to end our discussion with a few concluding remarks. First, one natural generalization we can consider is to extend the above discussion to the cases containing SO / Sp gauge groups. This can be done via orientifolding, that is, by including orientifold planes in the setup of section 2 in the spirit of [3]. In the $\mathcal{N}=2$ cases we expect no subtleties, but in the $\mathcal{N}=1$ cases with twists with $d_a=0$ some caution is needed [3] due to the subtleties discussed in [22, 23].

Another point we would like to comment on is the following. Recently, in the brane world context [24]–[40], it was pointed out in [38, 39] that the Einstein-Hilbert term is generically expected to be induced via loop corrections on a brane as long as the brane world-volume theory is not conformal. Subsequently, it was argued in [40] that this effect should arise in the context of non-conformal gauge theories from D3-brane, in particular, this is expected to be the case in theories discussed in this paper. It would be interesting to see whether the brane-bulk duality can be used for simplifying computation of such corrections on the D3-branes.

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A. Brane-bulk couplings

In section 5 we used various brane-bulk couplings in the $\mathbb{Z}_2$ orbifold examples. These couplings can be computed within the boundary state formalism, where one computes the couplings of the boundary states to the twisted closed string states in the presence of non-trivial D-brane field backgrounds [41]–[44]. Here, however, since we only need the couplings relevant for the one-loop corrections in the gauge theory language, we will take a shortcut and deduce these couplings using the annulus amplitude in the presence of D-brane field backgrounds.

[26] On the other hand, it is not completely clear how to check whether there are no non-perturbative corrections to the Kähler potential.

The annulus amplitude in the light-cone gauge is given by (here R is the generator of the $\mathbb{Z}_2$ orbifold twist):

$$\mathcal{C} = \int_0^\infty \frac{dt}{t} \,\mathrm{Tr}\left[\frac{1+R}{2}\frac{1-(-1)^F}{2}e^{-2\pi t L_0}\right] . \tag{A.1}$$

Here we assume that we have a non-trivial *constant* background gauge field along the brane. The annulus amplitude $\mathcal{C}$ then is almost the same as in the case without the background field, which we discussed in section 2, with the difference that some of the open string oscillator modings are modified [41]. For our purposes here it will suffice to consider the background field of the form ($F_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, is the D3-brane gauge field strength):

$$F_{23} = BQ\,, \tag{A.2}$$

where Q is a Cartan generator of the $\mathrm{U}(N)$ gauge group, which we will take to be hermitean. Let q_α, $\alpha = 1, \ldots, N$, be the eigenvalues of Q. It is convenient to introduce complex combinations of the world-sheet bosonic and fermionic degrees of freedom corresponding to the directions $\mu = 2, 3$ (so that instead of two real world-sheet bosons and two real world-sheet fermions corresponding to these directions we have one complex world-sheet boson and one complex world-sheet fermion). Then the modings of these complex world-sheet degrees of freedom are modified as follows (modings of other world-sheet degrees of freedom are unchanged):

$$r \to r + \Delta_{\alpha\beta}\,, \tag{A.3}$$

where $\Delta_{\alpha\beta}$ is given by:

$$\Delta_{\alpha\beta} = \frac{1}{\pi}\left[\arctan\left(2\pi\alpha' B q_\alpha\right) - \arctan\left(2\pi\alpha' B q_\beta\right)\right] . \tag{A.4}$$

Here the indices α, β label the two ends of the open string.

The cylinder amplitude is given by [42]:

$$\mathcal{C} = \int_0^\infty \frac{dt}{t}\frac{1}{8\pi^2\alpha' t}\sum_{\alpha,\beta}\frac{(q_\alpha - q_\beta)B}{2\pi}\left[U_{\alpha\beta} + T_{\alpha\beta}\right] e^{-\frac{t}{2\pi\alpha'}X^2_{\alpha\beta}}\,, \tag{A.5}$$

where $X^2_{\alpha\beta}$ is square of the distance (in the two real extra directions untouched by the orbifold) between the D3-branes labeled by α and β. The factor of $8\pi^2\alpha' t$ in the denominator comes from the bosonic zero modes in the directions $\mu = 0, 1$. The factor $(q_\alpha - q_\beta)B/2\pi$ can be understood from the requirement that in the $B \to 0$ limit we must reproduce the corresponding answer (see below). In particular, the untwisted character $U_{\alpha\beta}$ and the twisted character $T_{\alpha\beta}$ are given by[27] (for simplicity

[27] Here we should note that there is some freedom in choosing the phases multiplying the terms containing $Z^{-1/2}_{-1/2}$. Thus, for instance, we could have chosen the phase of the last term in the square brackets in (A.6) as -1 instead of $+1$. Note, however, that these phases do not affect our results here as the terms they multiply are vanishing ($Z^{-1/2}_{-1/2} = 0$).

the indices α, β are not shown):

$$U = \frac{1}{4\eta^6(q) Z_{-1/2}^{-1/2+\Delta}} \Big[Z_0^\Delta [Z_0^0]^3 - Z_{-1/2}^\Delta [Z_{-1/2}^0]^3 - - Z_0^{-1/2+\Delta} [Z_0^{-1/2}]^3 + Z_{-1/2}^{-1/2+\Delta} [Z_{-1/2}^{-1/2}]^3 \Big], \tag{A.6}$$

$$T = \frac{1}{\eta^2(q) Z_{-1/2}^{-1/2+\Delta} [Z_0^{-1/2}]^2} \Big[Z_0^\Delta Z_0^0 [Z_{-1/2}^0]^2 - Z_{-1/2}^\Delta Z_{-1/2}^0 [Z_0^0]^2 - - Z_0^{-1/2+\Delta} Z_{-1/2}^0 [Z_{-1/2}^{-1/2}]^2 + Z_{-1/2}^{-1/2+\Delta} Z_{-1/2}^{-1/2} [Z_{-1/2}^0]^2 \Big]. \tag{A.7}$$

Here the characters Z_u^v are the usual complex fermion characters ($-1/2 \leq v < 1/2$, $q \equiv \exp(-2\pi t)$):

$$Z_u^v \equiv q^{\frac{1}{2}v^2 - \frac{1}{24}} \prod_{m=1}^{\infty} \left(1 + q^{m+v-\frac{1}{2}} e^{-2\pi i u}\right) \left(1 + q^{m-v-\frac{1}{2}} e^{2\pi i u}\right). \tag{A.8}$$

In particular, note that the character $Z_{-1/2}^{-1/2+\Delta}$ in the denominators in (A.6) and (A.7) in the $B \to 0$ limit becomes:

$$Z_{-1/2}^{-1/2+\Delta} \to 2\pi t \Delta \frac{1}{\eta^2(q)}. \tag{A.9}$$

Combining this with the aforementioned factor $(q_\alpha - q_\beta)B/2\pi$ gives the following contribution

$$\frac{(q_\alpha - q_\beta)B}{4\pi^2 \Delta_{\alpha\beta} t} \frac{1}{\eta^2(q)} \to \frac{1}{8\pi^2 \alpha' t} \frac{1}{\eta^2(q)}, \tag{A.10}$$

which precisely corresponds to the bosonic zero modes plus oscillators in the $\mu = 2, 3$ directions in the absence of the gauge field background.

Here we would like to extract couplings of the massless closed string states to the D-branes. This can be done by extracting the leading behavior of the annulus amplitude for $t \to 0$. Using modular transformation properties of the above characters, we obtain:

$$U \sim \frac{t^3}{\sin(\pi\Delta)} \left\{ \left[2 - \sin^2(\pi\Delta)\right]\Big|_{\text{NS-NS}} - 2\cos(\pi\Delta)|_{\text{R-R}} \right\},$$
$$T \sim \frac{2t}{\sin(\pi\Delta)} \left\{ 1|_{\text{NS-NS}} - \cos(\pi\Delta)|_{\text{R-R}} \right\}, \tag{A.11}$$

where individual contributions due to the NS-NS and R-R exchanges are shown.

Here we are interested in the brane-bulk couplings involving at most quadratic terms in the gauge field strength $F_{\mu\nu}$. Then the relevant terms in the small B limit are given by:

$$U \sim \frac{2t^3}{\pi\Delta} \left[1 - \frac{1}{3}(\pi\Delta)^2\right] \left\{ 1|_{\text{NS-NS}} - 1|_{\text{R-R}} \right\},$$
$$T \sim \frac{t}{\pi\Delta} \left[1 - \frac{1}{3}(\pi\Delta)^2\right] \left\{ \left[2 + (\pi\Delta)^2\right]\Big|_{\text{NS-NS}} - 2|_{\text{R-R}} \right\}. \tag{A.12}$$

Using the fact, which follows from (A.4), that to the relevant order

$$\frac{1}{\pi\Delta_{\alpha\beta}}\left[1-\frac{1}{3}(\pi\Delta_{\alpha\beta})^2\right]=\frac{1+(2\pi\alpha' B)^2 q_\alpha q_\beta}{(2\pi\alpha' B)(q_\alpha-q_\beta)}\,, \tag{A.13}$$

we obtain the following massless untwisted respectively massless twisted closed string contributions into the annulus amplitude

$$\begin{aligned}
\widetilde{\mathcal{C}}_U &= \frac{1}{(4\pi^2\alpha')^2}\left\{1|_{\text{NS-NS}}-1|_{\text{R-R}}\right\}\sum_{\alpha,\beta}\left[1+(2\pi\alpha' B)^2 q_\alpha q_\beta\right]\int_0^\infty dtt e^{-\frac{t}{2\pi\alpha'}X^2_{\alpha\beta}}\,,\\
\widetilde{\mathcal{C}}_T &= \frac{1}{2(4\pi^2\alpha')^2}\sum_{\alpha,\beta}\left\{\left[2+(2\pi\alpha' B)^2\left(q_\alpha^2+q_\beta^2\right)\right]\Big|_{\text{NS-NS}}-\left[2+2(2\pi\alpha' B)^2 q_\alpha q_\beta\right]\Big|_{\text{R-R}}\right\}\times\\
&\quad\times\int_0^\infty \frac{dt}{t}e^{-\frac{t}{2\pi\alpha'}X^2_{\alpha\beta}}\,.
\end{aligned} \tag{A.14}$$

Note that the integrals in these expressions are related to the corresponding euclidean propagators $\Delta_6(X^2_{\alpha\beta})$ and $\Delta_2(X^2_{\alpha\beta})$:

$$\Delta_d(y^2)\equiv\int\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot y}}{p^2}=\frac{1}{4\pi^{d/2}}\int_0^\infty dss^{\frac{d-4}{2}}e^{-sy^2}\,. \tag{A.15}$$

here $y^2\equiv y^iy^i$, $i=1,\ldots,d$, and p^i are the momenta corresponding to the coordinates y^i.

As we discussed in section 5, the relevant part of the classical action is given by (note that for the field of the form (A.2) terms containing $F\wedge F$ are vanishing):

$$\begin{aligned}
S=&-\int_{\text{D3}}\Big(a\,\mathrm{Tr}\left[F^{\mu\nu}F_{\mu\nu}\right]+b\sigma\,\mathrm{Tr}\left[\gamma_R F^{\mu\nu}F_{\mu\nu}\right]+\mathrm{Tr}(\gamma_R)\widehat{c}\sigma+d\epsilon^{\mu\nu\sigma\rho}C_{\mu\nu}\,\mathrm{Tr}\left[\gamma_R F_{\sigma\rho}\right]\Big)-\\
&-\int_{\text{D3}\times\mathbb{R}^2}\left(\frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma+\frac{1}{12}H^{\mu\nu\sigma}H_{\mu\nu\sigma}\right).
\end{aligned} \tag{A.16}$$

Here σ is a twisted NS-NS scalar, while $C_{\mu\nu}$ is a twisted two-form (whose field strength is $H_{\mu\nu\sigma}$). The twisted Chan-Paton matrix $\gamma_R=\mathbb{1}_N$. Note, however, that for generic values of $X_{\alpha\beta}$ the $\mathrm{U}(N)$ gauge group is broken down to $\mathrm{U}(1)^N$. So the D3-branes are not necessarily coincident in the two transverse dimensions untouched by the orbifold action.

From the tree-level action (A.16) we obtain the following twisted massless contributions quadratic in the gauge field strength (here we are using $F^{\mu\nu}F_{\mu\nu}=2B^2Q^2$):

$$\widetilde{\mathcal{C}}'_T=2b\widehat{c}\sum_{\alpha,\beta}2B^2\left(q_\alpha^2+q_\beta^2\right)\Delta_2(X^2_{\alpha\beta})\Big|_\sigma-8d^2\sum_{\alpha,\beta}2B^2q_\alpha q_\beta\Delta_2(X^2_{\alpha\beta})\Big|_{C_{\mu\nu}}\,. \tag{A.17}$$

Note that this expression contains an overall factor of 2 from exchanging the two ends of the string as we are discussing an *oriented* string theory. Comparing this expression with (A.14), we obtain:

$$b\widehat{c}=2d^2=\frac{1}{8\pi}\,. \tag{A.18}$$

The coupling $\widehat{c}$ can be determined in a similar fashion.

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